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Three Parameters Fuzzy Sets Framework: Applications in Non Associative Algebraic Structures*

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Abstract. In this paper, we introduce the concept of three parameters-fuzzy sets. We define three parameters fuzzy ideals in non associative algebraic structures namely

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Abel Grassman groupoids (AG-groupoids). We explored several properties of an intra-regular AG-groupoid using three parameters-fuzzy AG-subgroupoids and three parameters-fuzzy right ideals.

Keywords: AG-groupoid; Completely regular; $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy ideals.

1. Introduction

Dealing with imprecise information is a common task and challenge in everyday life, as uncertainty is inevitably involved in every real world system. Two most important types of approach one which is first of all represented by theoretical construction based on suitable and coherent logical reference models are constructed to control, predict, and diagnose such systems, and so uncertainty should be properly incorporated into system description. The second which builds on multivariate statistical methodology (discriminate analysis, factor analysis, cluster analysis, correspondence analysis) and attempts to aggregate, within reasonable dimensions, the basic information dispersed in considerably extensive indicator vectors. For a long time dealing with uncertain information was a big challenge. In practice we often experience those real situations that are impossible to describe accurately. If we assign a truth-value of one to the element that is included in the set, and a truth value comparable to zero to such an element that lies outside the set, we create the range of two valued logic. This sort of logic assumes that precise symbols must be employed, and it is therefore not applicable to the real existence but only to an imagined existence. If we consider the characteristic features of real world systems, we will conclude that real situations are very often uncertain or vague in a number of ways. If the information demanded by a system is lacking, the future state of such a system may not be known completely. This type of uncertainty has been handled by probability theories and statistics, and it is called stochastic uncertainty. The vagueness, concerning the description of the semantic meaning of the events, phenomena, or statements themselves, is called fuzziness.

Until the 1960s, uncertainty was considered solely in terms of probability theory and understood as randomness. This seemingly unambiguous connection between uncertainty and probability was paralleled by several mathematical theories, distinct from probability theory, which are able to characterize situations under uncertainty.

The creator of fuzzy set theory, L.A. Zadeh, referred to the last hypothesis when he wrote: "As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until the threshold is reached beyond which precision and significance become almost mutually exclusive characteristics"

Formal control logic is based in the teachings of Aristotle, where an element either is or is not a member of a particular set, L.A. Zadeh was one of those who investigated alternative forms of data classification. The result of this investigation was the introduction of fuzzy sets and fuzzy theory at the University of California Berkeley in 1965 [15]. Fuzzy logic, a more generalized data set, allows for a "class" with continuous membership gradations. This form of classification with degrees of membership offers a much wider scope of applicability, especially in control applications. Fuzzy logic techniques have been applied to a wide range of systems, with many electronic control systems in the automotive industry, such as automatic transmission, engine control and antilock braking systems. This important concept has opened up new insights and application in a wide range of scientific field and plays an important role for solving real life problems involving ambiguities. Rosenfeld in 1971, introduced the concept of fuzzy set theory in groups [11]. Mordeson et al. [7] have discussed the vast field of fuzzy semigroups, where theoretical exploration of fuzzy semigroups and the applications of fuzzy set are used in fuzzy coding, fuzzy automata and finite state machines. The theory of soft sets (see [3, 4]) has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc.

Fuzzy set theory on semigroups has already been developed. In [8] Murali initiated the notion of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set was defined in [10]. These two concepts played a vital role in producing different types of fuzzy subgroups. On bases of these ideas Bhakat and Das [1, 2] introduced the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In [12] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [13] semigroups were characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals.

An AG-groupoid is a non-associative algebraic structure lies in between a groupoid and a commutative semigroup. Although it is non-associative, some times it possesses some interesting properties of a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup. Now our aim is to find out some logical investigations for intra-regular AG-groupoids using the new generalized concept of fuzzy sets.

In this paper, we introduced some new types of fuzzy ideals namely $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$ -fuzzy right ideals in AG-groupoids and develop some new results. We give some characterizations for intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideals.

2. Preliminaries

A groupoid $(S, .)$ is called an AG-groupoid (LA-semigroup in some articles [9]), if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [6], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all a, b, c, $d \in S$.

If an AG-groupoid contains a left identity, the following law holds

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

Let S be an AG-groupoid. By an AG-subgroupoid of S we mean a nonempty subset A of S such that $A^2 \subseteq A$.

A left (right) ideal of S is a nonempty subset I of S such that $SI \subseteq I$ $(IS \subseteq I)$. By a two-sided ideal or simply ideal, we mean a nonempty subset of S which is both a left and a right ideal of S.

A nonempty subset A of an AG-groupoid S is called semiprime of S if $a^2 \in A$ implies $a \in A$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of $S, f \leq g$ means that $f(x) \leq g(x)$ and $(f \cap g)(x) = f(x) \wedge g(x)$ for all x in S.

Let f and g be any fuzzy subsets of an AG-groupoid S. Then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\} & \text{if there exist } b, c \in S, \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions can be found in [7]:

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all $x, y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy ideal of S if it is both a fuzzy left and fuzzy right ideal of S.

Let $\Im(X)$ denote the collection of all fuzzy subsets of an AG-groupoid S with a left identity.

Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover $S \circ S = S$.

Definition 2.1. *A fuzzy subset f of X of the form,*

$$f(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by (x, r), where $r \in (0, 1]$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-Fuzzy Sets

In this section, we define three parameters fuzzy sets which are generalizations of $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy sets and named them $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$ −fuzzy sets or triplet fuzzy sets. Let $\gamma, \delta, \eta \in [0,1]$ be such that $\gamma < \eta < \delta$. For any $B \subseteq A$, let $\chi_{\gamma B}^{\Gamma}$ where $\Gamma = (\gamma, \delta)$ be a fuzzy subset of X such that $\chi_{\gamma B}^{\Gamma}(x) \geq \min\{\delta, \eta\} = \eta$ for all $x \in B$ and $\chi_{\gamma B}^{\Gamma}(x) \leq \gamma$ other wise. Clearly, $\chi_{\gamma B}^{\Gamma}$ is the characteristic function of B if $\gamma = 0$ and $\eta = 1$.

For fuzzy point x_r and a fuzzy subset f of X, we see that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
(2) $x_r q_\delta^\eta f$ if $f(x) + r + \eta > 3\delta$.
(3) $x_r \in_\gamma \vee q_\delta^\eta f$ if $x_r \in_\gamma f$ or $x_r q_\delta^\eta f$.

Now we introduce a new relation on $\Im(X)$, denoted by "$\subseteq \vee q_{(\gamma,\delta)}^\eta$", as follows:

For any $f, g \in \Im(X)$, by $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$ we mean that $x_r \in_r f$ implies $x_r \in_\gamma \vee q_\delta^\eta g$, for all $x \in X$ and $r \in (\gamma, \eta]$. Moreover f and g are said to be (γ^η, δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$ and $g \subseteq \vee q_{(\gamma,\delta)}^\eta f$.

Definition 3.1. [2] *A fuzzy subset λ of a group G is said to be a fuzzy subgroup of G if for all $x, y \in G$,*

(1) $\lambda(xy) \succeq M(\lambda(x), \lambda(y))$ *and*
(2) $\lambda(x^{-1}) \succeq \lambda(x)$.

Definition 3.2. [2] *A fuzzy subset λ of a group G is said to be an $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$ and $t, r \in (0,1]$.*

(1) $x_t, y_r \in \lambda$ *implies* $(xy)_{M(t,r)} \in \vee q\lambda$ *and*
(2) $\lambda(x^{-1}) \succeq \lambda(x)$.

Definition 3.3. [2] *A fuzzy subset λ of a group G is said to be an $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$ and $t, r \in (0,1]$.*

(1) $x_t, y_r \in \lambda$ *implies* $(xy)_{M(t,r)} \in \vee q\lambda$ *and*
(2) $x_t \in \lambda$ *implies* $x_t^{-1}) \in \vee q\lambda$.

Lemma 3.4. *Let f and g be a fuzzy subset of $\Im(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$ if and only if* $\max\{g(x), \gamma\} \geq \min\{f(x), \delta, \eta\}$ *for all $x \in X$.*

Proof. For fuzzy subset f and g we have $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$. Assume that there exist $x \in X$ and $t \in (\gamma, \eta]$ such that $\max\{g(x), \gamma\} < t \leq \min\{f(x), \delta, \eta\}$. Then $max\{g(x), \gamma\} < t > \gamma$. This implies that $g(x) < t > \gamma$, which further implies that $x_t \overline{\in}_\gamma g$, also $g(x) + t + \eta \leq 3\delta$, implies that $x_t \overline{q_\delta^\eta} g$, therefore $x_t \overline{\in_\gamma \vee q_\delta^\eta} g$. From $\min\{f(x), \delta, \eta\} \geq t$, implies that $f(x) \geq t > \gamma$, which implies that $x_t \in_\gamma f$. But $x_t \overline{\in_\gamma \vee q_\delta^\eta} g$ a contradiction to the definition of $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$. Hence

$$\max\{g(x),\gamma\} \geq \min\{f(x),\delta,\eta\} \text{ for all } x \in X.$$

Conversely, assume that there exist $x \in X$ and $t \in (\gamma, \eta]$ such that

$$\max\{g(x),\gamma\} \geq \min\{f(x),\delta,\eta\}$$

Further let $x_t \in_\gamma f$ implies that $f(x) \geq t > \gamma$. Need to show that $x_t \in_\gamma \vee q_\delta^\eta g$. Now

$$\max\{g(x),\gamma\} \geq \min\{f(x),\delta,\eta\} \geq \min\{t,\delta,\eta\} = t$$

but $\max\{g(x),\gamma\} = g(x)$, therefore $g(x) \geq t > \gamma$, implies that $x_t \in_\gamma g$, which implies that $x_t \in_\gamma \vee q_\delta^\eta g$. Hence $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$. ■

Lemma 3.5. *Let f,g and $h \in \Im(X)$. If $f \subseteq \vee q_{(\gamma,\delta)}^\eta g$ and $g \subseteq \vee q_{(\gamma,\delta)}^\eta h$, then $f \subseteq \vee q_{(\gamma,\delta)}^\eta h$.*

The relation "$=_{(\gamma^\eta,\delta)}$" is equivalence relation on $\Im(X)$, Moreover, $f =_{(\gamma^\eta,\delta)} g$ if and only if $\max\{\min\{f(x),\delta\},\gamma\} = \max\{\min\{g(x),\delta\},\gamma\}$ for all $x \in X$.

Lemma 3.6. *Let A, B be any non-empty subset of an AG-groupoid S with left identity. Then we have*

(1) *$A \subseteq B$ if only if $\chi_{\gamma A}^\Gamma \subseteq_{(\gamma^\eta,\delta)} \chi_{\gamma B}^\Gamma$, where $\Gamma = (\gamma, \delta)$ and γ,δ,$\eta \in [0, 1]$.*
(2) *$\chi_{\gamma A}^\Gamma \cap \chi_{\gamma B}^\Gamma =_{(\gamma^\eta,\delta)} \chi_{\gamma(A\cap B)}^\Gamma$.*
(3) *$\chi_{\gamma A}^\Gamma \circ \chi_{\gamma B}^\Gamma =_{(\gamma^\eta,\delta)} \chi_{\gamma(AB)}^\Gamma$.*

Definition 3.7. *A fuzzy subset f of an AG-groupoid S with a left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, \eta]$, such that $x_t \in_\gamma f$, $y_t \in_\gamma f$ we have $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta^\eta$.*

Theorem 3.8. *Let f be a fuzzy subset of an AG-groupoid S with a left identity. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy AG-subgroupoid of S if and only if*

$$\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta,\eta\} \text{ where } \gamma,\delta \in [0,1].$$

Proof. Let f be a fuzzy subset of an AG-groupoid S which is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy AG-subgroupoid of S. Assume that there exist $x, y \in S$, and $t \in (\gamma, \eta]$, such that

$$\max\{f(xy),\gamma\} < t \leq \min\{f(x), f(y),\delta,\eta\}.$$

Then $\max\{f(xy),\gamma\} < t \geq \gamma$, implies that $f(xy) < t \geq \gamma$, which further implies that $(xy)_t \overline{\in_\gamma} f$, also $f(xy) + t + \eta \leq 3\delta$, implies that $(xy)_t \overline{q_\delta^\eta} f$, therefore $(xy)_t \overline{\in_\gamma \vee q_\delta^\eta} f$ also and $\min\{f(x), f(y),\delta,\eta\} \geq t$ implies that $f(x) \geq t > \gamma$,

$f(y) \geq t > \gamma$, which implies that $x_t \in_\gamma f$, $y_t \in_\gamma f$. But $(xy)_t \overline{\in_\gamma \vee q_\delta^\eta} f$, which contradicts the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta, \eta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, \eta]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$. Then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$. Now by hypothesis we have

$$\begin{aligned} \max\{f(xy), \gamma\} &\geq \min\{f(x), f(y), \delta, \eta\} \\ &\geq \min\{t, s, \delta, \eta\} \\ &> \min\{t, s\} > \gamma, \end{aligned}$$

but $\max\{f(xy), \gamma\} = f(xy)$, therefore $f(xy) \geq \min\{t, s\} > \gamma$, implies that $(xy)_{\min\{t,s\}} \in_\gamma f$,which implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta^\eta f$. Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ which means that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta^\eta f$ for all $x, y \in S$,therefore f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy AG-subgroupoid of S. ■

Definition 3.9. *A fuzzy subset f of an AG-groupoid S with a left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right (respt left) ideal of S if for all $x, y \in S$ and $t \in (\gamma, \eta]$ such that $x_t \in_\gamma f$ we have $(xy)_t \in_\gamma \vee q_\delta^\eta f$ resp. $y_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta^\eta f$.*

Theorem 3.10. *A fuzzy subset f of an AG-groupoid S with a left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta, \eta\} \text{ for all } x,y \in S.$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of S. Suppose there are $x, y \in S$ and $t \in (\gamma, \eta]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), \delta, \eta\} \text{ for some } x, y \in S.$$

Then $\max\{f(xy), \gamma\} < t \geq \gamma$. This implies that $f(xy) < t \geq \gamma$, which further implies that $(xy)_t \overline{\in_\gamma} f$, also $f(xy) + t + \eta \leq 3\delta$, implies that $(xy)_t \overline{q_\delta^\eta} f$, therefore $(xy)_t \overline{\in_\gamma \vee q_\delta^\eta} f$. From $\min\{f(x), \delta, \eta\} \geq t$, implies that $f(x) \geq t > \gamma$, which implies that $x_t \in_\gamma f$. But $(xy)_t \overline{\in_\gamma \vee q_\delta^\eta}$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta, \eta\} \text{ for all } x,y \in S.$$

Conversely, assume that there are $x, y \in S$ and $t \in (\gamma, \eta]$ such that

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta, \eta\}.$$

Further let $x_t \in_\gamma f$. This implies that $f(x) \geq t > \gamma$. Need to show that $(xy)_t \in_\gamma \vee q_\delta^\eta f$. Now

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta, \eta\} \geq \min\{t, \delta, \eta\} = t,$$

but $\max\{f(xy),\gamma\} = f(xy)$, therefore $f(xy) \geq t > \gamma$, implies that $(xy)_t \in_\gamma f$, which implies that $(xy)_t \in_\gamma \vee q_\delta^\eta f$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of S. ∎

Example 3.11. Let $S = \{1, 2, 3\}$ and define the binary operation $\circ$ as follows:

$\circ$	1	2	3
1	2	3	3
2	2	2	2
3	2	2	2

Then $(S, \circ)$ is an AG-groupoid. Let us define a fuzzy subset f on S follows:

$$f(x) = \begin{cases} 0.29 & \text{if } x = 1 \\ 0.4 & \text{if } x = 2 \\ 0.22 & \text{if } x = 3 \end{cases}$$

Then, we have

(1) f is an $\left(\in_{0.2}, \in_{0.2} \vee q_{0.3}^{0.21}\right)$-fuzzy subgroupoid of S.

(2) f is not an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$-fuzzy subgroupoid of S, because

$$0.22 = \max\{f(1 \circ 2), 0.2\} < \min\{f(1), 0.3\} = 0.29$$

Example 3.12. Consider the AG-groupoid defined by the following multiplication table on $S = \{1, 2, 3, \}$:

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then clearly $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.21 & \text{if } x = 1 \\ 0.1 & \text{if } x = 2 \\ 0.22 & \text{if } x = 3 \end{cases}$$

Then, we have

(1) f is an $\left(\in_{0.01}, \in_{0.01} \vee q_{0.2}^{0.1}\right)$-fuzzy right ideal.

(2) f is not an $(\in_{0.01}, \in_{0.01} \vee q_{0.2})$-fuzzy right ideal, because

$$0.1 = \max\{f(3 \circ 2), 0.01\} < \min\{f(3), 0.2\} = 0.2$$

(3) f is not a fuzzy right ideal because

$$f(3 \circ 2) < f(3)$$

Lemma 3.13. *R is a right ideal of an AG-groupoid S if and only if $\chi^{\Gamma}_{\gamma R}$ is an $(\in_{\gamma}, \in_{\gamma} \vee q^{\eta}_{\delta})$-fuzzy right ideal of S.*

Proof. (i) Let $x, y \in R$. It means that $xy \in R$. Then $\chi^{\Gamma}_{\gamma R}(xy) \geq \delta + \eta > \eta$, $\chi^{\Gamma}_{\gamma R}(x) > \eta$ and $\chi^{\Gamma}_{\gamma R}(y) > \eta$ but $\gamma < \eta < \delta$. Thus

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \chi^{\Gamma}_{\gamma R}(xy) \text{ and } \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \delta.$$

Hence $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

(ii) Let $x \notin R$, $y \in R$.

Case 1. If $xy \notin R$, then $\chi^{\Gamma}_{\gamma R}(x) \leq \gamma$, $\chi^{\Gamma}_{\gamma R}(y) > \eta$ and $\chi^{\Gamma}_{\gamma R}(xy) < \gamma$. Therefore

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \gamma \text{ and } \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \chi^{\Gamma}_{\gamma R}(x).$$

Hence $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

Case 2. If $xy \in R$, then $\chi^{\Gamma}_{\gamma R}(xy) > \eta$, $\chi^{\Gamma}_{\gamma R}(x) \leq \gamma$ and $\chi^{\Gamma}_{\gamma R}(y) > \eta$. Therefore

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \chi^{\Gamma}_{\gamma R}(xy)\, and \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \chi^{\Gamma}_{\gamma R}(x)$$

Hence $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

(iii) Let $x \in R$, $y \notin R$. Then $xy \in R$. Thus $\chi^{\Gamma}_{\gamma R}(xy) > \delta$, $\chi^{\Gamma}_{\gamma R}(y) \leq \gamma$ and $\chi^{\Gamma}_{\gamma R}(x) > \eta$. Thus

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \chi^{\Gamma}_{\gamma R}(xy) \text{ and } \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \delta.$$

Hence $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

(iv) Let $x, y \notin R$. Then

Case 1. Assume that $xy \notin R$. Then by definition we get $\chi^{\Gamma}_{\gamma R}(xy) \leq \gamma$, $\chi^{\Gamma}_{\gamma R}(y) \leq \gamma$ and $\chi^{\Gamma}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \chi^{\Gamma}_{\gamma R}(xy) \text{ and } \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \delta.$$

Hence $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

Case 2. Assume that $xy \in R$. Then by definition we get $\chi^{\Gamma}_{\gamma R}(xy) > \eta$, $\chi^{\Gamma}_{\gamma R}(y) \leq \gamma$ and $\chi^{\Gamma}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} = \chi^{\Gamma}_{\gamma R}(xy) \text{ and } \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\} = \chi^{\Gamma}_{\gamma R}(x).$$

Therefore $\max\left\{\chi^{\Gamma}_{\gamma R}(xy), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(x), \delta, \eta\right\}$.

Conversely, let $rs \in RS$, where $r \in R$ and $s \in S$. By hypothesis $\max\left\{\chi^{\Gamma}_{\gamma R}(rs), \gamma\right\} \geq \min\left\{\chi^{\Gamma}_{\gamma R}(r), \delta, \eta\right\}$. Since $r \in R$, thus $\chi^{\Gamma}_{\gamma R}(r) > \eta$ which implies that $\min\left\{\chi^{\Gamma}_{\gamma R}(r), \delta, \eta\right\} = \eta$. Thus

$$\max\left\{\chi^{\Gamma}_{\gamma R}(rs), \gamma\right\} \geq \eta$$

This implies that $\chi^{\Gamma}_{\gamma R}(rs) \geq \eta$ which implies that $rs \in R$. Hence R is a right ideal of S. ∎

Definition 3.14. *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime if $x_t^2 \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta^\eta f$ for all $x \in S$ and $t \in (\gamma, \eta]$.*

Example 3.15. Consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with the following multiplication table

$\circ$	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	4	5	6	3
4	1	2	3	4	5	6
5	1	2	6	3	4	5
6	1	2	5	6	3	2

Clearly $(S, \circ)$ is intra-regular because $1 = (3 \circ 1^2) \circ 1$, $2 = (2 \circ 2^2) \circ 2$, $3 = (4 \circ 3^2) \circ 6$, $4 = (4 \circ 4^2) \circ 4$, $5 = (6 \circ 5^3) \circ 3$, $6 = (5 \circ 6^2) \circ 5$.

Define a fuzzy subset f on S as given:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.32 & \text{if } x = 2 \\ 0.4 & \text{if } x = 3 \\ 0.42 & \text{if } x = 4 \\ 0.5 & \text{if } x = 5 \\ 0.2 & \text{if } x = 6 \end{cases}$$

Then f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.22}^{0.11})$-fuzzy semiprime of S.

Theorem 3.16. *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime if and only if* $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta, \eta\}$.

Proof. Let f be $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime. Assume that there exists $a \in S$ and $t \in (\gamma, \eta]$, such that

$$\max\{f(a), \gamma\} < t \leq \min\{f(a^2), \delta, \eta\}.$$

Then $\max\{f(a), \gamma\} < t$. This implies that $f(a) < t > \gamma$, which implies that $a_t \overline{\in_\gamma f}$. Now since $\eta < \delta < t$, so $f(a) + t + \eta < 3\delta$. Thus $a_t \overline{\in_\gamma \vee q_\delta^\eta} f$. Also since $\min\{f(a^2), \delta, \eta\} \geq t$, implies that $f(a^2) \geq t > \gamma$, this further implies that $a_t^2 \in_\gamma f$. Thus by definition of $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime $a_t \in_\gamma \vee q_\delta^\eta f$ which is contradiction to $a_t \overline{\in_\gamma \vee q_\delta^\eta} f$. Hence

$$\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta, \eta\}, \text{ for all } a \in S.$$

Conversely, assume that there exist $a \in S$ and $t \in (\gamma, \eta]$ such that $a_t^2 \in_\gamma f$. Then $f\left(a^2\right) \geq t > \gamma$, thus

$$\max\{f(a), \gamma\} \geq \min\left\{f\left(a^2\right), \delta, \eta\right\} \geq \min\{t, \delta\} = t$$

implies that $\max\{f(a), \gamma\} \geq t$, but $\max\{f(a), \gamma\} = f(a)$, therefore $f(a) \geq t > \gamma$, implies that $a_t \in_\gamma f$, which implies that $a_t \in_\gamma \vee q_\delta^\eta f$. Hence f is $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime. ∎

Theorem 3.17. *For a right ideal R of an AG-groupoid S with a left identity, the following statements are equivalent:*

(1) *R is semiprime.*
(2) *$\chi_{\gamma R}^\Gamma$ is $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.*

Proof. (1) ⇒ (2) Let R be a semiprime ideal of an AG-groupoid S. Let a be an arbitrary element of S such that $a \in R$. Then $a^2 \in R$. Hence $\chi_{\gamma R}^\Gamma(a) \geq \eta$ and $\chi_{\gamma R}^\Gamma\left(a^2\right) \geq \eta$ which implies that $\max\left\{\chi_{\gamma R}^\Gamma(a), \gamma\right\} \geq \min\left\{\chi_{\gamma R}^\Gamma\left(a^2\right), \delta, \eta\right\}$.

Now let $a \notin R$. Since R is semiprime, we have $a^2 \notin R$. This implies that $\chi_{\gamma R}^\Gamma(a) \leq \gamma$ and $\chi_{\gamma R}^\Gamma\left(a^2\right) \leq \gamma$. Therefore, $\max\left\{\chi_{\gamma R}^\Gamma(a), \gamma\right\} \geq \min\left\{\chi_{\gamma R}^\Gamma\left(a^2\right), \delta, \eta\right\}$. Hence, $\max\left\{\chi_{\gamma R}^\Gamma(a), \gamma\right\} \geq \min\left\{\chi_{\gamma R}^\Gamma\left(a^2\right), \delta, \eta\right\}$ for all $a \in S$.

(2) ⇒ (1) Let $\chi_{\gamma R}^\Gamma$ be a fuzzy semiprime. If $a^2 \in R$, for some a in S, then $\chi_{\gamma R}^\Gamma\left(a^2\right) \geq \eta$. Since $\chi_{\gamma R}^\Gamma$ is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime, we have $\max\left\{\chi_{\gamma R}^\Gamma(a), \gamma\right\} \geq \min\left\{\chi_{\gamma R}^\Gamma\left(a^2\right), \delta, \eta\right\}$. Therefore $\max\left\{\chi_{\gamma R}^\Gamma(a), \gamma\right\} \geq \eta$. But $\gamma < \eta < \delta$, so $\chi_{\gamma R}^\Gamma(a) \geq \eta$. Thus $a \in R$. Hence R is semiprime. ∎

4. Intra-Regular AG-Groupoids

An element a of an AG-groupoid S is called intra-regular if there exist $x, y \in S$ such that $a = \left(xa^2\right)y$. S is called intra-regular, if every element of S is intra-regular.

Theorem 4.1. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*
(2) *For a right ideal R of an AG-groupoid S, $R \subseteq R^2$ and R is semiprime.*
(3) *For an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)}^\eta f \circ f$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.*

Proof. (1) ⇒ (3) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of an intra-regular AG-groupoid S with a left identity. Since S is intra-regular, for any a in S there exist x,y in S such that $a = \left(xa^2\right) y$. Now

$$\begin{aligned} a &= \left(xa^2\right) y = \left(x\left(aa\right)\right) y = \left(a\left(xa\right)\right) y = \left(y\left(xa\right)\right) a \\ &= \left(\left(uv\right)\left(xa\right)\right) a \text{ where } y = uv \\ &= \left(\left(ax\right)\left(vu\right)\right) a = \left(a\left(vu\right)\right)\left(ax\right) \end{aligned}$$

For any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned} \max\left\{\left(f \circ f\right)\left(a\right), \gamma\right\} &= \max\left\{\bigvee_{a=pq}\left\{\left\{f\left(p\right) \vee f\left(q\right)\right\}, \gamma\right\}\right\} \\ &\geq \max\left\{\min\left\{\left\{f\left(a\left(vu\right)\right), f\left(ax\right)\right\}, \gamma\right\}\right\} \\ &\geq \min\left\{\max\left\{f\left(a\left(vu\right)\right), \gamma\right\}, \max\left\{f\left(ax\right), \gamma\right\}\right\} \\ &\geq \min\left\{\min\left\{f\left(a\right), \delta, \eta\right\}, \min\left\{f\left(a\right), \delta, \eta\right\}\right\} \\ &= \min\left\{f\left(a\right), \delta, \eta\right\} \end{aligned}$$

Thus by Lemma 3.4, $f \subseteq \vee q^{\eta}_{(\gamma,\delta)} f \circ f$. Now to show that f is an $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist u, v in S such that $y = uv$. Thus

$$\begin{aligned} a &= \left(xa^2\right) y = \left(xa^2\right)\left(uv\right) = \left(vu\right)\left(a^2 x\right) \\ &= a^2\left(\left(vu\right) x\right) = a^2 s, \text{ where } \left(\left(vu\right) x\right) = s. \end{aligned}$$

Then

$$\max\left\{f\left(a\right), \gamma\right\} = \max\left\{f\left(a^2 s\right), \gamma\right\} \geq \min\left\{f\left(a^2\right), \delta, \eta\right\}$$

Hence, f is an $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy semiprime.

(3) ⇒ (2) Let R be any right ideal of an AG-groupoid S. By (3), $\chi^{\Gamma}_{\gamma R}$ is semiprime and by Theorem 3.17, R is semiprime. Now using Lemma 3.6, and (3), we get

$$\chi^{\Gamma}_{\gamma R} = \chi^{\Gamma}_{\gamma R \cap R} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma R} \cap \chi^{\Gamma}_{\gamma R} \subseteq \vee q^{\eta}_{(\gamma,\delta)} \chi^{\Gamma}_{\gamma R} \circ \chi^{\Gamma}_{\gamma R} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma R^2}$$

Hence by (1), we get $R \subseteq R^2$.

(2) ⇒ (1) Since Sa^2 is a right ideal containing a^2, using (2) we get

$$a \in Sa^2 \subseteq \left(Sa^2\right)^2 = \left(Sa^2\right)\left(Sa^2\right) \subseteq \left(Sa^2\right) S.$$

Hence S is intra-regular. ■

Theorem 4.2. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *For any right ideal R and any subset A of an AG-groupoid S, $R \cap A \subseteq RA$ and R is a semiprime ideal.*

(3) *For any $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy subset g, $f \cap g \subseteq \vee q_{(\gamma,\delta)}^\eta f \circ g$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.*

Proof. (1) $\Rightarrow$ (3) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy subset of an intra-regular AG-groupoid S. Since S is intra-regular, for any a in S there exist x,y in S such that $a = \left(xa^2\right) y$. Now using medial and paramedial law, we get

$$\begin{aligned} a &= \left(xa^2\right) y = [(x\,(aa))\,y] = [(a\,(xa))\,y] \\ &= [y\,(xa)]\,a, \\ y\,(xa) &= \left[y\left\{x\left(\left(xa^2\right)y\right)\right\}\right] = \left[y\left\{\left(xa^2\right)(xy)\right\}\right] = \left[\left(xa^2\right)\left(xy^2\right)\right] \\ &= \left[\left(y^2x\right)\left(a^2x\right)\right] = \left[a^2\left(y^2x^2\right)\right]. \end{aligned}$$

Thus

$$\begin{aligned} a &= \left(a^2t\right)a, \text{since } \left(y^2x^2\right) = t, \\ \max\{(f \circ g), \gamma\} &= \max\left\{\bigvee_{a=bc}\{f(b) \wedge g(c)\}, \gamma\right\} \\ &\geq \max\left\{\min\left\{\left\{f\left(a^2t\right), g(a)\right\}, \gamma\right\}\right\} \\ &= \min\left\{\max\left\{f\left(a^2t\right), \gamma\right\}, \max\{g(a), \gamma\}\right\} \\ &\geq \min\{\min\{f(a), \delta, \eta\}, \min\{g(a), \delta, \eta\}\} \\ &= \min\{f(a), g(a), \delta, \eta\} \\ &= \min\{(f \cap g)(a), \delta, \eta\}. \end{aligned}$$

By Lemma 3.4, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)}^\eta g \circ f$. Next we show that f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist u, v in S such that $y = uv$. Thus

$$\begin{aligned} a &= \left(xa^2\right) y = \left(xa^2\right)(uv) = (vu)\left(a^2x\right) \\ &= a^2\,((vu)\,x) = a^2s, \text{ where } ((vu)\,x) = s. \end{aligned}$$

Then

$$\max\{f(a), \gamma\} = \max\left\{f\left(a^2s\right), \gamma\right\} \geq \min\left\{f\left(a^2\right), \delta, \eta\right\}$$

Hence, f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.

(3) $\Rightarrow$ (2) Let R be a right ideal and A be a subset of S. By Lemma 3.6, and (3), we get

$$\chi_{\gamma(R\cap A)}^\Gamma =_{(\gamma^\eta,\delta)} \chi_{\gamma R}^\Gamma \cap \chi_{\gamma A}^\Gamma \subseteq \vee q_{(\gamma,\delta)}^\eta \chi_{\gamma R}^\Gamma \circ \chi_{\gamma A}^\Gamma =_{(\gamma^\eta,\delta)} \chi_{\gamma RA}^\Gamma.$$

By Lemma 3.6, $R \cap A \subseteq RA$. Next to show that R is a semiprime. By (3), $\chi^{\Gamma}_{\gamma R}$ is semiprime and by Theorem 3.17, R is semiprime.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal containing a^2. By (2), it is semiprime. Therefore

$$a \in Sa^2 \cap Sa \subseteq \left(Sa^2\right)(Sa) \subseteq \left(Sa^2\right) S.$$

Hence S is intra-regular. ■

Theorem 4.3. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *For any right ideal R and any subset A of an AG-groupoid S, $R \cap A \subseteq AR$ and R is a semiprime ideal.*

(3) *For any $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy subset g, $f \cap g \subseteq \vee q^\eta_{(\gamma,\delta)} g \circ f$ and f is an $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy semiprime.*

Proof. (1) $\Rightarrow$ (3) Let f be an $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy subset of an intra-regular AG-groupoid S. Since S is intra-regular, for any a in S there exist x,y in S such that $a = \left(xa^2\right) y$. Now using medial and paramedial law, we get

$$\begin{aligned} a &= \left(xa^2\right) y = \left(xa^2\right)(uv) \text{ since } y = uv \\ &= (vu)\left(a^2 x\right) = a^2\left((vu)x\right) = (aa)\left((vu)x\right) \\ &= \left(x(vu)\right)(aa) = a\left(\left(x(vu)\right)a\right) = a(za) \text{ since } z = x(vu). \end{aligned}$$

Now

$$\begin{aligned} za &= z\left(\left(xa^2\right)y\right) = \left(xa^2\right)(zy) = (yz)\left(a^2x\right) = a^2\left((yz)x\right) \\ &= (aa)\left((yz)x\right) = \left(x(yz)\right)(aa) = a\left(\left(x(yz)\right)a\right) = at. \end{aligned}$$

where $t = \left(x(yz)\right)a$. Therefore, we have

$$\begin{aligned} a &= a(at), \\ \max\left\{(f \circ g), \gamma\right\} &= \max\left\{\bigvee_{a=bc}\left\{g(b) \wedge f(c)\right\}, \gamma\right\} \\ &\geq \max\left\{\min\left\{\left\{g(a), f(at)\right\}, \gamma\right\}\right\} \\ &= \min\left\{\max\left\{g(a), \gamma\right\}, \max\left\{f(at), \gamma\right\}\right\} \\ &\geq \min\left\{\min\left\{g(a), \delta, \eta\right\}, \min\left\{f(a), \delta, \eta\right\}\right\} \\ &= \min\left\{g(a), f(a), \delta, \eta\right\} \\ &= \min\left\{(f \cap g)(a), \delta, \eta\right\}. \end{aligned}$$

By Lemma 3.4, we have $f \cap g \subseteq \vee q^\eta_{(\gamma,\delta)} g \circ f$. Next we show that f is an $(\in_\gamma, \in_\gamma \vee q^\eta_\delta)$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist u, v in

S such that $y = uv$. Thus

$$\begin{aligned} a &= \left(xa^2\right) y = \left(xa^2\right)(uv) = (vu)\left(a^2x\right) \\ &= a^2\left((vu)x\right) = a^2 s, \text{ where } ((vu)x) = s. \end{aligned}$$

Then

$$\max\{f(a), \gamma\} = \max\left\{f\left(a^2 s\right), \gamma\right\} \geq \min\left\{f\left(a^2\right), \delta, \eta\right\}$$

Hence, f is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.

(3) $\Rightarrow$ (2) Let R be a right ideal and A be a subset of S. By Lemma 3.6, and (3), we get

$$\chi^\Gamma_{\gamma(R\cap A)} = \chi^\Gamma_{\gamma(A\cap R)} =_{(\gamma^\eta,\delta)} \chi^\Gamma_{\gamma A} \cap \chi^\Gamma_{\gamma R} \subseteq \vee q^\eta_{(\gamma,\delta)} \chi^\Gamma_{\gamma A} \circ \chi^\Gamma_{\gamma R} =_{(\gamma^\eta,\delta)} \chi^\Gamma_{\gamma AR}.$$

By Lemma 3.6, $R \cap A \subseteq AR$. Next to show that R is a semiprime. By (3), $\chi^\Gamma_{\gamma R}$ is semiprime and by Theorem 3.17, R is semiprime.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal containing a^2. By (2), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa^2 \cap Sa \subseteq (Sa)\left(Sa^2\right) = \left(a^2 S\right)(aS) = [(aa)(SS)](aS) \\ &= [(SS)(aa)](aS) \subseteq \left(Sa^2\right) S \end{aligned}$$

Hence S is intra-regular. ■

Theorem 4.4. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *For any right ideal R and any subset A of an AG-groupoid S, $A \cap R \subseteq AR$ and R is a semiprime ideal.*

(3) *For any $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy subset f and any $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal g of S, $f \cap g \subseteq \vee q^\eta_{(\gamma,\delta)} f \circ g$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.*

Proof. (1) $\Rightarrow$ (3) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy subset and g be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is an intra-regular it follow that for any a in S there exist x,y in S such that $a = \left(xa^2\right) y$. Now using medial and paramedial law, we get

$$\begin{aligned} a &= \left(xa^2\right) y = \left(xa^2\right)(uv) \text{ since } y = uv \\ &= (vu)\left(a^2 x\right) = a^2\left((vu)x\right) = (aa)\left((vu)x\right) \\ &= (x(vu))(aa) = a\left((x(vu))a\right) = a(za) \text{ since } z = x(vu), \\ za &= z\left(\left(xa^2\right)y\right) = \left(xa^2\right)(zy) = (yz)\left(a^2 x\right) = a^2\left((yz)x\right). \end{aligned}$$

Thus

$$a = a\left(a^2 v\right), \text{ where } (yz)x = v.$$

For any a in S there exist s and t in S such that $a = st$. Then

$$\begin{aligned} \max\{(f\circ g)(a),\gamma\} &= \max\left\{\bigvee_{a=st}\{f(s)\wedge g(t)\},\gamma\right\} \\ &\geq \max\left\{\min\left\{\left\{f(a), g\left(a^2v\right)\right\},\gamma\right\}\right\} \\ &= \min\left\{\max\{f(a),\gamma\},\max\left\{g\left(a^2v\right),\gamma\right\}\right\} \\ &\geq \min\{\min\{f(a),\delta,\eta\},\min\{g(a),\delta,\eta\}\} \\ &= \min\{f(a), g(a),\delta,\eta\} \\ &= \min\{(f\cap g)(a),\delta,\eta\}. \end{aligned}$$

Thus by Lemma 3.4, $f\cap g \subseteq \vee q^{\eta}_{(\gamma,\delta)} g\circ f$.

(3) $\Rightarrow$ (2) Let R be a right ideal and A be any subset of S. By Lemma 3.6, and (3), we get

$$\chi^{\Gamma}_{\gamma(A\cap R)} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma A}\cap\chi^{\Gamma}_{\gamma R} \subseteq \vee q^{\eta}_{(\gamma,\delta)}\chi^{\Gamma}_{\gamma A}\circ\chi^{\Gamma}_{\gamma R} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma AR}.$$

By Lemma 3.6, $A\cap R\subseteq AR$.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal containing a^2. By (2), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa\cap Sa^2 \subseteq (Sa)\left(Sa^2\right) = \left(a^2S\right)(aS) \subseteq \left(a^2S\right)S \\ &= \left(a^2(SS)\right)S = \left((SS)a^2\right)S = \left(Sa^2\right)S. \end{aligned}$$

Hence S is intra-regular. ∎

Theorem 4.5. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *For any subsets A,B and for any right ideal R of S, $A\cap B\cap R\subseteq (AB)R$ and R is a semiprime ideal.*

(3) *For any $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy subsets f,g and any $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy right ideal of h, $f\cap g\cap h\subseteq \vee q^{\eta}_{(\gamma,\delta)}((f\circ g)\circ h)$ and h is an $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy semiprime ideal of S.*

Proof. (1) $\Rightarrow$ (2) Let f,g be $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy subsets and h be an $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular it follow that for any a in S there exist x,y in S such that $a=\left(xa^2\right)y$. Now using medial, paramedial laws, we get

$$\begin{aligned} a &= \left(xa^2\right)y = [(uv)(aa)]y, \text{ where } x=uv \\ &= [(aa)(vu)]y = [y(vu)](aa) = a[\{y(vu)\}a] \\ &= a(ta), \text{ where } t = y(vu). \\ ta &= t\left[\left(xa^2\right)y\right] = \left(xa^2\right)(ty) = \left[(yt)\left(a^2x\right)\right] = (aa)[(yt)x] \\ &= [x(yt)](aa) = a[\{x(yt)\}a] = a(za), \text{ where } z = x(yt). \\ za &= z\left[\left(xa^2\right)y\right] = \left(xa^2\right)(zy) = (yz)\left(a^2x\right) = a^2[(yz)x] \\ &= a^2[(yz)x] = a^2w, \text{ where } w=(yz)x. \end{aligned}$$

Thus $a = a\left[a\left(a^2w\right)\right] = a\left[a^2\left(aw\right)\right] = a^2\left[a\left(aw\right)\right]$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned}
&\max\left\{\left(\left(f\circ g\right)\circ h\right)\left(a\right),\gamma\right\}\\
=\ &\max\left\{\bigvee_{a=bc}\left\{\left(f\circ g\right)\left(b\right)\wedge h\left(c\right)\right\},\gamma\right\}\\
\geq\ &\max\left\{\min\left\{\left(f\circ g\right)\left(aa\right),h\left(a\left(aw\right)\right)\right\},\gamma\right\}\\
\geq\ &\max\left\{\min\left\{f\left(a\right),g\left(a\right),h\left(a\left(aw\right)\right)\right\},\gamma\right\}\\
=\ &\min\left\{\max\left\{f(a),\gamma\right\},\max\left\{g(a),\gamma\right\},\max\left\{h\left(a\left(aw\right)\right),\gamma\right\}\right\}\\
\geq\ &\min\left\{\min\left\{f(a),\delta,\eta\right\},\min\left\{g(a),\delta,\eta\right\},\min\left\{h\left(a\left(aw\right)\right),\delta,\eta\right\}\right\}\\
=\ &\min\left\{\min\left\{f(a),g(a),h(a)\right\},\delta,\eta\right\}\\
=\ &\min\left\{\left(f\cap g\cap h\right)\left(a\right),\delta,\eta\right\}.
\end{aligned}$$

Thus by Lemma 3.4, $f\cap g\cap h\subseteq \vee q^{\eta}_{(\gamma,\delta)}\left(\left(f\circ g\right)\circ h\right)$. Next we show that h is $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist u, v in S such that $y = uv$. Thus

$$\begin{aligned}
a\ &=\ \left(xa^2\right)y = \left(xa^2\right)\left(uv\right) = \left(vu\right)\left(a^2x\right)\\
&=\ a^2\left(\left(vu\right)x\right) = a^2s,\text{ where } s = \left(vu\right)x.
\end{aligned}$$

Then

$$\max\left\{h(a),\gamma\right\}\ =\ \max\left\{h\left(a^2s\right),\gamma\right\}\geq \min\left\{h(a^2),\delta,\eta\right\}.$$

Hence, h is an $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy semiprime.

(3) $\Rightarrow$ (2) Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.6 and (3), we get

$$\chi^{\Gamma}_{\gamma(A\cap B)\cap R} =_{(\gamma^\eta,\delta)} \chi^{\Gamma}_{\gamma A}\cap\chi^{\Gamma}_{\gamma B}\cap\chi^{\Gamma}_{\gamma R}\subseteq \vee q^{\eta}_{(\gamma,\delta)}\left(\chi^{\Gamma}_{\gamma A}\circ\chi^{\Gamma}_{\gamma B}\right)\circ\chi^{\Gamma}_{\gamma R} =_{(\gamma^\eta,\delta)} \chi^{\Gamma}_{\gamma(AB)R}.$$

By Lemma 3.6, we get $(A\cap B)\cap R\subseteq (AB)R$. Since R be any right ideal of an AG-groupoid S. By (3), $\chi^{\Gamma}_{\gamma R}$ is semiprime and by Theorem 3.17, R is semiprime.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal of an AG-groupoid S containing a^2. By (2), it is semiprime. Thus by (2), we get

$$a\in Sa\cap Sa\cap Sa^2\subseteq\left[\left(Sa\right)\left(Sa\right)\right]\left(Sa^2\right) = \left[\left(SS\right)\left(aa\right)\right]\left(Sa^2\right)\subseteq\left(Sa^2\right)S.$$

Hence S is intra-regular. ■

Theorem 4.6. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *For any subsets A,B and for any right ideal R of S,* $A \cap R \cap B \subseteq (AR)B$ *and R is a semiprime ideal.*

(3) *For any* $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$*-fuzzy subsets f,h and any* $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$*-fuzzy right ideal of g,* $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}^\eta ((f \circ g) \circ h)$ *and g is an* $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$*-fuzzy semiprime ideal of S.*

Proof. (1) $\Rightarrow$ (3) Let f, h be $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy subsets and g be an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular it follow that for any a in S there exist x,y in S such that $a = (xa^2)\, y$. Now using medial, paramedial laws, we get

$$\begin{aligned} a &= (xa^2)\, y = [(a(xa))y] = [(y(xa))a], \\ y(xa) &= y[x((xa^2)y)] = y[(xa^2)\,(yx)] = [(xa^2)\,(xy^2)] = (y^2x)\,(a^2x) \\ &= a^2\,(y^2x^2) = (aa)\,(y^2x^2) = (x^2y^2)\,(aa) = a\,[(x^2y^2)\,a] \\ (x^2y^2)\,a &= (x^2y^2)\,[(xa^2)\,y] = (xa^2)\,[(x^2y^2)\,y] \\ &= [y\,(x^2y^2)]\,(a^2x) = a^2\,[\{(y\,(y^2x^2))\}\,x] \end{aligned}$$

Thus $a = [a\,(a^2v)]\,a$, where $\{y\,(y^2x^2)\}\,x = v$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned} & \max\{((f \circ g) \circ h)(a), \gamma\} \\ =\ & \max\left\{\bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\ \geq\ & \max\{\min\{(f \circ g)(a(a^2v)), h(a)\}, \gamma\} \\ \geq\ & \max\{\min\{f(a), g(a^2v), h(a)\}, \gamma\} \\ =\ & \min\{\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}, \max\{h(a), \gamma\}\} \\ \geq\ & \min\{\min\{f(a), \delta, \eta\}, \min\{g(a), \delta, \eta\}, \min\{h(a), \delta, \eta\}\} \\ =\ & \min\{\min\{f(a), g(a), h(a)\}, \delta, \eta\} \\ =\ & \min\{(f \cap g \cap h)(a), \delta, \eta\}. \end{aligned}$$

Thus by Lemma 3.4, $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}^\eta ((f \circ g) \circ h)$. Next we show that g is $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist u, v in S such that $y = uv$. Thus

$$\begin{aligned} a &= (xa^2)\, y = (xa^2)\,(uv) = (vu)\,(a^2x) \\ &= a^2\,((vu)\,x) = a^2 s, \text{ where } s = (vu)\,x. \end{aligned}$$

Then

$$\max\{g(a), \gamma\} = \max\{g(a^2s), \gamma\} \geq \min\{g(a^2), \delta, \eta\}.$$

Hence, g is an $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy semiprime.

(3) $\Rightarrow$ (2) Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.6, and (iii), we get

$$\chi^{\Gamma}_{\gamma(A\cap R)\cap B} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma A} \cap \chi^{\Gamma}_{\gamma R} \cap \chi^{\Gamma}_{\gamma B} \subseteq \vee q^{\eta}_{(\gamma,\delta)} \left(\chi^{\Gamma}_{\gamma A} \circ \chi^{\Gamma}_{\gamma R}\right) \circ \chi^{\Gamma}_{\gamma B} =_{(\gamma^{\eta},\delta)} \chi^{\Gamma}_{\gamma(AR)B}.$$

By Lemma 3.6, we get $(A \cap R) \cap B \subseteq (AR)\, B$. Since R is a right ideal of an AG-groupoid S. By (iii), $\chi^{\Gamma}_{\gamma R}$ is semiprime and by Theorem 3.17, R is semiprime.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (2), we get

$$\begin{aligned} a &\in Sa \cap Sa^2 \cap Sa \subseteq \left[(Sa)\left(Sa^2\right)\right](Sa) \subseteq [S\left(Sa^2\right)]S \\ &= [S\left(Sa^2\right)](SS) = (SS)\left[\left(Sa^2\right)S\right] = S[\left(Sa^2\right)S] \\ &= \left(Sa^2\right)(SS) = \left(Sa^2\right)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 4.7. *Let S be an AG-groupoid with a left identity. Then the following conditions are equivalent:*

(1) *S is intra-regular.*
(2) *For any subsets A,B and for any right ideal R of S, $R \cap A \cap B \subseteq (RA)\, B$ and R is a semiprime ideal.*
(3) *For any $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy subsets g, h and any $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy right ideal of f, $f \cap g \cap h \subseteq \vee q^{\eta}_{(\gamma,\delta)} ((f \circ g) \circ h)$ and f is an $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy semiprime ideal of S.*

Proof. (1) $\Rightarrow$ (3) Let f be an $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy right ideal and g, h be $(\in_\gamma, \in_\gamma \vee q^{\eta}_{\delta})$-fuzzy subsets of an intra-regular AG-groupoid S. Now since S is intra-regular it follow that for any a in S there exist x,y in S such that $a = \left(xa^2\right)y$. Now using medial, and paramedial law, we get

$$\begin{aligned} a &= \left(xa^2\right)y = [(uv)(aa)]\,y, \text{ where } x = uv \\ &= [(aa)(vu)]\,y = [y\,(vu)]\,(aa) = (aa)\,[(vu)\,y] = [\{(vu)\,y\}\,a]\,a \\ &= (ta)\,a, \text{ where } t = (vu)\,y. \\ ta &= t\left[\left(xa^2\right)y\right] = \left(xa^2\right)(ty) = \left[(yt)\left(a^2x\right)\right] = (aa)\,[(yt)\,x] = [\{(yt)\,x\}\,a]\,a \\ &= (za)\,a, \text{ where } z = (yt)\,x, \text{and } za = z\left[\left(xa^2\right)y\right] = \left(xa^2\right)(zy) = (yz)\left(a^2x\right) \\ &= a^2\,[(yz)\,x] = a^2w, \text{ where } w = (yz)\,x. \end{aligned}$$

Thus $a = \left[\left(a^2w\right)a\right]a$. For any a in S there exist b and c in S such that

$a = bc$. Then

$$\begin{aligned}
&\max\{((f\circ g)\circ h)(a),\gamma\}\\
&= \max\left\{\bigvee_{a=bc}\{(f\circ g)(b)\wedge h(c)\},\gamma\right\}\\
&\geq \max\left\{\min\left\{(f\circ g)\left(\left(a^2w\right)a\right),h(a)\right\},\gamma\right\}\\
&\geq \max\left\{\min\left\{f\left(a^2w\right),g(a),h(a)\right\},\gamma\right\}\\
&= \min\left\{\max\left\{f\left(a^2w\right),\gamma\right\},\max\{g(a),\gamma\},\max\{h(a),\gamma\}\right\}\\
&\geq \min\{\min\{f(a),\delta,\eta\},\min\{g(a),\delta,\eta\},\min\{h(a),\delta,\eta\}\}\\
&= \min\{\min\{f(a),g(a),h(a)\},\delta,\eta\}\\
&= \min\{(f\cap g\cap h),\delta,\eta\}.
\end{aligned}$$

Thus by Lemma 3.4, $f\cap g\cap h\subseteq \vee q^{\eta}_{(\gamma,\delta)}((f\circ g)\circ h)$. Next we show that f is $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy semiprime. Since $S=S^2$, for each y in S there exist u,v in S such that $y=uv$. Thus

$$\begin{aligned}
a &= \left(xa^2\right)y=\left(xa^2\right)(uv)=(vu)\left(a^2x\right)\\
&= a^2((vu)x)=a^2s, \text{ where } s=(vu)x.
\end{aligned}$$

Then

$$\begin{aligned}
\max\{f(a),\gamma\} &= \max\left\{f\left(a^2s\right),\gamma\right\}\\
&\geq \min\left\{f(a^2),\delta,\eta\right\}.
\end{aligned}$$

Hence, f is an $(\in_\gamma,\in_\gamma\vee q^{\eta}_{\delta})$-fuzzy semiprime.

(3) $\Rightarrow$ (2) Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.6, and (3), we get

$$\begin{aligned}
\chi^{\Gamma}_{\gamma(R\cap A)\cap B} &= {}_{(\gamma^\eta,\delta)}\chi^{\Gamma}_{\gamma R}\cap\chi^{\Gamma}_{\gamma A}\cap\chi^{\Gamma}_{\gamma B}\\
&\subseteq \vee q^{\eta}_{(\gamma,\delta)}\left(\chi^{\Gamma}_{\gamma R}\circ\chi^{\Gamma}_{\gamma A}\right)\circ\chi^{\Gamma}_{\gamma B}\\
&= {}_{(\gamma^\eta,\delta)}\chi^{\Gamma}_{\gamma(R\cap A)\cap B}.
\end{aligned}$$

Hence by Lemma 3.6, we get $(R\cap A)\cap B\subseteq(R\cap A)\cap B$.

(2) $\Rightarrow$ (1) Sa^2 is a right ideal of an AG-groupoid S containing a^2. By (2), it is semiprime. Thus we get

$$\begin{aligned}
a &\in Sa^2\cap Sa\cap Sa\subseteq\left[\left(Sa^2\right)(Sa)\right]Sa\subseteq\left[\left(Sa^2\right)S\right]S\\
&= (SS)\left(Sa^2\right)=(SS)[(SS)(aa)]=(SS)[(aa)(SS)]\\
&= (SS)\left(a^2S\right)=\left(Sa^2\right)S.
\end{aligned}$$

Hence S is intra-regular. ■

Conclusion. In this paper, we characterized intra-regular AG-groupoids with a left identity using the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideals. We discussed several important features of an intra-regular AG-groupoid by using the $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy AG-subgroupoid and $(\in_\gamma, \in_\gamma \vee q_\delta^\eta)$-fuzzy right ideals. This study can give a new direction for applications of fuzzy set theory in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

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