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On Pseudo-Slant Submanifolds of Conformal Kenmotsu Manifolds

Venkatesha
Department of Mathematics, Kuvempu University, Shankaraghatta-577451, Shivamogga, Karnataka, India
Email: vensmath@gmail.com

N. Srikantha
Department of Mathematics, M.S. Ramaiah Institute of Techonology (Affiliated to VTU), Bengaluru, India
Email: srikantha087@gmail.com

M.S. Siddesha
Department of Mathematics, Jain University, Bengaluru-562112, India
Email: s.siddu2010@gmail.com

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Abstract. In this paper, we study the pseudo-slant submanifolds of conformal Kenmotsu manifolds. Some results on submanifolds of conformal Kenmotsu manifolds with parallel canonical structures are obtained. Finally we discuss integrability of the distributions embeded in the definition of pseudo-slant submanifolds of conformal Kenmotsu manifolds.

Keywords: Conformal Sasakian manifolds; Integrability; Parallel; Pseudo-slant submanifolds.

1. Introduction

A $2n$-dimensional Kähler manifold M with complex structure J and Hermitian metric g is said to be globally conformal Kähler manifolds if there exist a function $f : M \to \mathbb{R}$, such that the metric $exp(f)g$ is also a Kählerian. Libermann

[15] was the first to study this. Later Visman [19] proved the necessary and sufficient condition for a locally conformal Kähler manifold to be a Kähler. It is also considered as one of the sixteen classes of almost Hermitian manifolds classified by Banaru [4]. Abedi introduced the conformal Sasakian manifolds [1] and studied the submanifolds of conformal Sasakian manifolds. Since Kenmotsu manifolds are contact manifolds which are not Sasakian, E. Abedi and R. Abedi extended the concept of conformal manifolds to conformal Kenmotsu manifolds [2]. Later Abedi studied different submanifolds of conformal manifolds [3].

On the other side, to study the geometry of unknown manifolds geometers introduced the concept of embedding the unknown manifolds with rather known manifolds and studied the geometry parallelly. This initiated the study on theory of submanifolds and now it has wide range of applications in physics and mathematics. The study of slant submanifolds has played an important role in the study of spaces. This study was initiated by Chen [10, 9] on complex manifolds. As slant submanifolds are the generalization of invariant and anti-invariant submanifolds, many geometers has shown interest on this study. Lotta [16] introduced the concept of slant immersions in to an almost contact metric manifold. Carriazo introduced another new class of submanifolds called hemi-slant submanifolds (it is also called as anti-slant or pseudo-slant submanifold) [8]. Later many geometers (see [13], [11], [12], [14]) studied pseudo-slant submanifolds on various contact manifolds. In 2015, Tastan studied the hemi-slant submanifolds of a locally conformal Kähler manifold [17]. It is very interesting that those theorems or results of Kähler manifolds can be applied to contact manifolds. In this connection Venkatesha studied the pseudo-slant submanifolds of conformal Sasakian manifolds [18]. In this paper we extended the study of pseudo-slant submanifolds to conformal Kenmotsu manifolds.

The current paper is organized as follows: Section 2 deals with the basic definitions and results on conformal Kenmotsu manifolds and submanifolds. We give definition of pseudo-slant submanifolds and some important results in Section 3. Parallelism of the canonical structures of the submanifolds of conformal Kenmotsu manifolds are discussed in Section 3. The last section is devoted to integrability of the distributions embedded with the definition of pseudo-slant submanifolds of the conformal Kenmotsu manifolds.

2. Preliminaries

$2m+1$-dimensional differentiable manifold $\widetilde{M}$ is said to be contact manifold if a global 1-form η satisfies $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\widetilde{M}$. If $\widetilde{M}^{2m+1}$ with an almost contact metric structure (ϕ, ξ, η, g) satisfies (see [5, 6]):

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{1}$$

$$\phi\xi = 0, \ \eta \circ \phi = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(\phi X, Y) = -\,g(X, \phi Y), \quad g(X, \xi) = \eta(X), \tag{4}$$

for any vector fields on $\widetilde{M}$. Then $\widetilde{M}$ is called almost contact metric manifold, where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and g is a Riemannian metric. Let Φ be the fundamental 2-form on $\widetilde{M}$ defined by $\Phi(X,Y) = g(X, \phi Y) = -\Phi(Y, X)$. Now if $\Phi = d\eta$ then almost contact metric structure becomes contact metric structure.

Further a almost contact metric manifold $\widetilde{M}^{2m+1}$ is said to be Kenmotsu manifolds if

$$(\widetilde{\nabla}_X \phi)Y =, \ \widetilde{\nabla}_X \xi = -\phi X.$$

Let $\widetilde{M}$ be a smooth manifold. $(\widetilde{M}^{2m+1}, \phi, \xi, \eta, g)$ is called a conformal Kenmotsu manifold if [2]

$$\widetilde{g} = \exp(f) g, \ \widetilde{\phi} = \phi, \ \widetilde{\eta} = (\exp(f))^{1/2}\eta, \ \widetilde{\xi} = (\exp(-f))^{1/2}\xi.$$

Let $\widetilde{\nabla}$ and $\overline{\nabla}$ be the connections of $\widetilde{M}$ with respect to metrics $\widetilde{g}$ and g respectively, and are related by

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \frac{1}{2}\left\{\omega(X)Y + \omega(Y)X - g(X,Y)\zeta\right\}, \tag{5}$$

where ω is global 1-form defined by $\omega(X) = X(f)$ and ζ is Lee vector field metrically equivalent to ω i.e., $g(\zeta, X) = \omega(X)$.

Further for a conformal Kenmotsu manifold we have

$$\begin{aligned}(\overline{\nabla}_X \phi)Y =& (\exp(f))^{1/2}\left\{-g(X, \phi Y)\xi - \eta(Y)\phi X\right\} \\ & - \frac{1}{2}\left\{\omega(\phi Y)X - \omega(Y)\phi X + g(X,Y)\phi\zeta - g(X, \phi Y)\zeta\right\},\end{aligned} \tag{6}$$

$$\overline{\nabla}_X \xi = (\exp(f))^{1/2}\left\{X - \eta(X)\xi\right\} + \frac{1}{2}\left\{\eta(X)\zeta - \omega(\xi)X\right\}. \tag{7}$$

Definition 2.1. *Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with Riemannian metric g. Then for all $X, Y \in TM$ and $V \in T^{\perp}M$ the Gauss and Weingarten formulas with respect to $\overline{\nabla}$ are given by*

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{9}$$

where ∇(respectively $\nabla^{\perp}$) is the induced Riemannian (respectively normal) connection in TM (respectively $T^{\perp}M$) with respect to $\overline{\nabla}$, A and h are the shape operator and second fundamental form related by

$$g(h(X,Y), V) = g(A_V X, Y). \tag{10}$$

A submanifold M is said to be totally umbilical if $h(X,Y) = g(X,Y)H$, where H is the mean curvature of M in $\widetilde{M}$. If $h = 0$ (equivalently $A_V = 0$) then M is called totally geodesic.

Let for any $X \in TM$ and $V \in T^{\perp}M$. We can write

$$\phi X = TX + NX, \tag{11}$$

$$\phi V = tV + nV, \tag{12}$$

where TX and NX (respectively tV and nV) are the tangential and normal component of ϕX (respectively ϕV). Using (1) in the above equations one can get

$$T^2 = -tN - I + \eta \circ \xi, \quad NT + nN = 0, \tag{13}$$

$$n^2 = -I - Nt, \quad Tt + tn = 0. \tag{14}$$

Now for $X \in TM$, we take

$$X = PX + QX + \eta(X)\xi, \tag{15}$$

such that P and Q are the projections on D_θ and $D^{\perp}$ respectively.

3. Pseudo-slant Submanifold of Conformal Kenmotsu Manifold

Now let us recall some definitions of classes of submanifolds. Let M be a submanifold. Then M is said to be

- (i) Invariant submanifold if T is identically zero in (11), i.e., $\phi X \in TM,\ \forall X \in TM$.
- (ii) Anti-invariant submanifold if N is identically zero in (11), i.e., $\phi X \in T^{\perp}M,\ \forall X \in TM$.
- (iii) Slant submanifold if there exists an angle $\theta(x) \in [0, \pi/2]$ between ϕX and TX for all non-zero vector X tangent to M at x called slant angle which is constant.
- (iv) Pseudo-slant submanifold if there exists distributions D_θ and $D^{\perp}$ such that
  - (a) TM admits orthogonal direct composition $TM = D_\theta \oplus D^{\perp} \oplus < \xi >$. D_θ is a slant distribution with slant angle $\theta \neq \pi/2$.
  - (b) $D^{\perp}$ is an anti-invariant distribution [13].

From the above definitions we can note that slant submanifold is the generalization of invariant (if $\theta = 0$) and anti-invariant (if $\theta = \pi/2$) submanifolds. A proper slant submanifold is neither invariant nor anti-invariant submanifold i.e., $\theta \in (0, \pi/2)$. Hence in general we have the following theorem which characterize slant submanifolds of almost contact metric manifolds;

Theorem 3.1. [7] *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is slant submanifold if and only if there exists a constant $\gamma \in [0,1]$ such that*

$$T^2 = -\gamma(I - \eta \otimes \xi), \tag{16}$$

furthermore, in this case, if θ is the slant angle of M, then $\gamma = \cos^2\theta$.

Corollary 3.2. [7] *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TX, TY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{17}$$

$$g(NX, NY) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{18}$$

Lemma 3.3. *Let M be a proper pseudo-slant submanifold of conformal Kenmotsu manifold $\widetilde{M}$. Then*

$$\phi D^{\perp} \perp ND_{\theta}. \tag{19}$$

Proof. Let $X \in D^{\perp}, Y \in D_{\theta}$. In view of (3) and (11) we get $g(\phi X, NY) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) = 0$. ∎

Lemma 3.4. *Let M be a pseudo-slant submanifold of conformal Kenmotsu manifold $\widetilde{M}$. Then*

$$TD^{\perp} = \{0\}, \tag{20}$$

$$TD_{\theta} = D_{\theta}. \tag{21}$$

Proof. (20) follows from (11). Now for $X \in D^{\perp}$ and $Y \in D_{\theta}$,

$$g(X, TY) = g(X, \phi Y) = -g(Y, \phi X) = 0.$$

The above equation shows that $TD_{\theta} \perp D^{\perp}$. Also we have $g(TY, \xi) = 0$ and from the fact that $TD_{\theta} \subseteq TM$ we can infer that $TD_{\theta} \subseteq D_{\theta}$. Now for $X \in D_{\theta}$, we have from (16) that

$$X = \frac{1}{\cos^2\theta}(\cos^2\theta) = \frac{1}{\cos^2\theta}(-T^2X) = -\frac{1}{\cos^2\theta}(T(TX)).$$

Hence we get $D_{\theta} \subseteq TD_{\theta}$. Thus we have (20). ∎

Let $\widetilde{M}$ be a conformal Kenmotsu manifold, M be a proper pseudo-slant submanifold of $\widetilde{M}$. We take ζ^T and $\zeta^{\perp}$ as tangential and normal parts of Lee vector field ζ, i.e.,

$$\zeta = \zeta^T + \zeta^{\perp}. \tag{22}$$

In view of (6), (8), (9), (11), (12) and the above equation we have the following lemma.

Lemma 3.5. *Let M be any submanifold of conformal Kenmotsu manifold $\widetilde{M}$. Then*

$$(\nabla_X T)Y = A_{NY}X + th(X,Y) + (\exp(f))^{1/2}\{g(X,\phi Y)\xi - \eta(Y)TX\} \tag{23}$$
$$-\frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX + g(X,Y)T\zeta^T + g(X,Y)t\zeta^\perp - g(X,TY)\zeta^T\},$$
$$(\nabla^\perp_X N)Y = -h(X,TY) + nh(X,Y) + (\exp(f))^{1/2}\{\eta(Y)NX\} \tag{24}$$
$$-\frac{1}{2}\{-\omega(Y)NX + g(X,Y)N\zeta^T + g(X,Y)n\zeta^\perp - g(X,TY)\zeta^\perp\},$$
$$(\nabla_X t)V = A_{nV}X - TA_VX - (\exp(f))^{1/2}\{g(X,tV)\xi\} \tag{25}$$
$$-\frac{1}{2}\{\omega(\phi V)X - \omega(V)TX - g(X,tV)\zeta^T\},$$
$$(\nabla^\perp_X n)V = -h(X,tV) - NA_VX + \frac{1}{2}\{\omega(V)NX + g(X,tV)\zeta^\perp\}, \tag{26}$$

for any $X,Y \in TM$ and $V \in T^\perp M$.

4. Parallelism of the Canonical Structures of the Submanifold of Conformal Kenmotsu Manifold

Theorem 4.1. *Let M be a submanifold of a conformal Kenmotsu manifold $\widetilde{M}$. Then T is parallel if and only if*

$$A_{NW}Y - A_{NY}W = (\exp(f))^{1/2}\{\eta(W)\phi Y + \eta(Y)TW\} - \frac{1}{2}\{\omega(\phi Y)W + \omega(Y)TW + g(T\zeta^T + t\zeta^\perp, W)Y - TYg(\zeta^T,W)\}, \tag{27}$$

for any $Y,W \in TM$.

Proof. Let $X,Y \in TM$. From (23) we have

$$0 = A_{NY}X + th(X,Y) + (\exp(f))^{1/2}\{g(X,\phi Y)\xi - \eta(Y)TX\}$$
$$-\frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX + g(X,Y)T\zeta^T + g(X,Y)t\zeta^\perp - g(X,TY)\zeta^T\}.$$

Taking inner product of this equation with $W \in TM$, we get

$$0 = g(A_{NY}X,W) + g(th(X,Y),W) + (\exp(f))^{1/2}\{g(X,\phi Y)\eta(W)$$
$$-\eta(Y)g(TX,W)\} - \frac{1}{2}\{\omega(\phi Y)g(X,W) - \omega(Y)g(TX,W)$$
$$+ g(X,Y)g(T\zeta^T + t\zeta^\perp, W) - g(X,TY)g(\zeta^T,W)\}.$$

Thus, using the condition for totally umbilical in the above equation we get (27).

Converse part is trivial. ■

Theorem 4.2. *Let M be a submanifold of a conformal Kenmotsu manifold $\widetilde{M}$. Then covariant derivative of T is skew-symmetric.*

Proof. Let $X, Y, W \in TM$. Using (23), (11), (12) and (22) we have

$$\begin{aligned}
g((\nabla_X T)Y, W) =& g(A_{NY}X, W) + g(th(X,Y), W) + (\exp(f))^{1/2}\{g(X, \phi Y)\eta(W) \\
&- \eta(Y)g(TX, W)\} - \frac{1}{2}\{\omega(\phi Y)g(X, W) - \omega(Y)g(TX, W) \\
&+ g(X,Y)g(T\zeta^T + t\zeta^\perp, W) - g(X, TY)g(\zeta^T, W)\} \\
=& g(h(X,W), NY) - g(h(X,Y), NW) \\
&+ (\exp(f))^{1/2}\{-g(\phi X, Y)\eta(W) + \eta(Y)g(X, \phi W)\} \\
&- \frac{1}{2}\{g(\zeta, \phi Y)g(X, W) - g(\zeta, Y)g(TX, W) \\
&+ g(X,Y)g(T\zeta^T + t\zeta^\perp, W) + g(TX, Y)g(\zeta, W)\}, \\
g((\nabla_X T)Y, W) =& - g(th(X,W), Y) - g(A_{NW}X, Y) \\
&- (\exp(f))^{1/2}\{\eta(Y)g(X, \phi W) - g(TX, Y)\eta(W)\} \\
&- \frac{1}{2}\{-g(T\zeta^T + t\zeta^\perp, Y)g(X, W) - g(\zeta, Y)g(TX, W) \\
&+ g(X,Y)g(T\zeta^T + t\zeta^\perp, W) + g(TX, Y)g(\zeta, W)\} \\
=& - g(th(X,W) + A_{NW}X + (\exp(f))^{1/2}\{g(X, \phi W)\xi \\
&- \eta(W)TX\} - \frac{1}{2}\{(T\zeta^T + t\zeta^\perp)g(X,W) + \zeta^T g(TX, W) \\
&+ Xg(\zeta, \phi W) - TXg(\zeta, W)\}, Y) = -g((\nabla_X T)W, Y).
\end{aligned}$$ ■

Theorem 4.3. *Let M be a submanifold of a conformal Kenmotsu manifold $\widetilde{M}$. Then N is parallel if and only if t is parallel.*

Proof. Let $X, Y \in TM$ and $V \in T^\perp M$. In view of (24), (11), (12), (22) and (25), we have

$$\begin{aligned}
& g((\nabla_X N)Y, V) \\
=& - g(h(X, TY), V) + g(nh(X,Y), V) \\
&+ (\exp(f))^{1/2}\{\eta(Y)g(NX, V)\} \\
&- \frac{1}{2}\{-\omega(Y)g(NX, V) + g(X,Y)g(N\zeta^T, V) \\
&+ g(X,Y)g(n\zeta^\perp, V) - g(X, TY)g(\zeta^\perp, V)\} \\
=& - g(A_V X, TY) - g(h(X,Y), nV) - (\exp(f))^{1/2}\{\eta(Y)g(X, tV)\} \\
&- \frac{1}{2}\{g(\zeta^\perp, Y)g(X, tV) - g(X,Y)g(\zeta^T, tV) - g(X,Y)g(\zeta^\perp, nV)
\end{aligned}$$

$$
\begin{aligned}
&+ g(TX, Y)g(\zeta, V)\}.\\
=\ & g(TA_V X, Y) - g(A_{nV}X, Y) - (\exp(f))^{1/2}\{\eta(Y)g(X, tV)\}\\
&- \frac{1}{2}\{g(\zeta, Y)g(X, tV) - g(X, Y)g(\zeta, \phi V) + g(TX, Y)\omega(V)\} \qquad (28)\\
&= -g(-TA_V X + A_{nV}X - (\exp(f))^{1/2}\{g(X, tV)\xi\}\\
&- \frac{1}{2}\{-\zeta g(X, tV) + X\omega(\phi V) - TX\omega(V)\}, Y)\\
&= g((\nabla_X t)V, Y).
\end{aligned}
$$

This completes our proof. ∎

Theorem 4.4. *Let M be a submanifold of conformal Kenmotsu manifold $\widetilde{M}$. Then N is parallel if and only if*

$$
\begin{aligned}
&A_{nV}Y + A_V TY\\
&= -(\exp(f))^{1/2}\{\eta(Y)tV\} + \frac{1}{2}\{\omega(\phi V)Y + \omega(V)TY - \omega(Y)tV\},
\end{aligned}
\tag{29}
$$

for any $Y \in TM$ and $V \in T^{\perp}M$.

Let $X, Y \in TM$, $V \in T^{\perp}M$ and N be parallel. From (28), we have

$$
\begin{aligned}
0 =& g((\nabla_X N)Y, V)\\
=& -g(A_V TY, X) - g(A_{nV}Y, X) - (\exp(f))^{1/2}\{\eta(Y)g(X, tV)\}\\
&- \frac{1}{2}\{\omega(Y)g(X, tV) - g(X, Y)\omega(\phi V) - g(X, TY)\omega(V)\}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
0 =& -g(A_V TY + A_{nV}Y + (\exp(f))^{1/2}\{\eta(Y)tV\}\\
&+ \frac{1}{2}\{\omega(Y)tV - \omega(\phi V)Y - \omega(V)TY\}, X).
\end{aligned}
$$

This proves our assertion.

Theorem 4.5. *Let M be a submanifold of conformal Kenmotsu manifold $\widetilde{M}$. Then covariant derivative of n is skew-symmetric.*

Proof. Let $X \in TM$ and $U, V \in T^{\perp}M$. Then from (26), (11), (12) and (22) we get

$$
g((\nabla_X n)V, U) = -g((\nabla_X n)U, V).
$$

Hence the covariant derivative of n is skew-symmetric. ∎

5. Integrability of the Distributions

Theorem 5.1. *Anti-invariant distribution $D^{\perp}$ of a pseudo-slant submanifold M of conformal Kenmotsu manifold $\widetilde{M}$ is integrable if and only if*

$$A_{NY}X - A_{NX}Y = \frac{1}{2}\{\omega(NY)X - \omega(NX)Y\}. \tag{30}$$

Proof. Let $X, Y \in D^{\perp}$. Consider

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= g(\nabla_Y \xi, X) - g(\nabla_X \xi, Y). \end{aligned}$$

From (7), we get

$$g([X,Y],\xi) = 0, \text{ for any } X, Y \in D^{\perp}. \tag{31}$$

Further, from (23) and (20) we have

$$\begin{aligned} -T\nabla_X Y - A_{NY}X - th(X,Y) =& (\exp(f))^{1/2}\{g(X,\phi Y)\xi - \eta(Y)TX\} \\ & - \frac{1}{2}\{\omega(NY)X + g(X,Y)(T\zeta^T + t\zeta^{\perp})\} \\ =& - \frac{1}{2}\{\omega(NY)X + g(X,Y)(T\zeta^T + t\zeta^{\perp})\}. \end{aligned} \tag{32}$$

By interchanging X and Y we get

$$-T\nabla_Y X - A_{NX}Y - th(X,Y) = -\frac{1}{2}\{\omega(NX)Y + g(X,Y)(T\zeta^T + t\zeta^{\perp})\}. \tag{33}$$

Using (32) and (33) and the fact that h is symmetric we get

$$T[X,Y] + A_{NY}X - A_{NX}Y = \frac{1}{2}\{\omega(NY)X - \omega(NX)Y\}.$$

Thus our assertion follows from (31) and (20). ∎

Theorem 5.2. *Slant distribution D_θ of a pseudo-slant submanifold M of conformal Kenmotsu manifold $\widetilde{M}$ is integrable if and only if*

$$\begin{aligned} &Q\{\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X + \frac{1}{2}\{\omega(\phi X)Y \\ &- \omega(\phi Y)X - \omega(X)TY + \omega(Y)TX\} + g(X,TY)\zeta^T\} = 0, \end{aligned} \tag{34}$$

for any $X, Y \in D_\theta$.

Proof. Let $X, Y \in D_\theta$. From (23) and (21) we get

$$\begin{aligned} \nabla_X TY - T\nabla_X Y =& A_{NY}X + th(X,Y) \\ & + (\exp(f))^{1/2}\{g(X,\phi Y)\xi\} - \frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX \\ & + g(X,Y)(T\zeta^T + t\zeta^{\perp}) - g(X,TY)\zeta^T\}. \end{aligned} \tag{35}$$

Interchanging X and Y in the above equation we get

$$\begin{aligned}\nabla_Y TX - T\nabla_Y X =& A_{NX}Y + th(X,Y) \\ &+ (\exp(f))^{1/2}\{g(Y,\phi X)\xi\} - \frac{1}{2}\{\omega(\phi X)Y - \omega(X)TY \\ &+ g(X,Y)(T\zeta^T + t\zeta^\perp) - g(Y,TX)\zeta^T\}.\end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned}T[X,Y] =& \nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X \\ &+ 2(\exp(f))^{1/2}\{g(X,\phi Y)\xi\} + \frac{1}{2}\{\omega(\phi X)Y - \omega(\phi Y)X \\ &- \omega(X)TY + \omega(Y)TX\} + g(X,TY)\zeta^T.\end{aligned} \tag{37}$$

Now applying Q (as defined in (15)) to the above equation we get,

$$\begin{aligned}QT[X,Y] =& Q\{\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X \\ &+ \frac{1}{2}\{\omega(\phi X)Y - \omega(\phi Y)X - \omega(X)TY + \omega(Y)TX\} + g(X,TY)\zeta^T\}.\end{aligned}$$

Thus we infer that D_θ is integrable if and only if (34) satisfies. ∎

Let M be a pseudo-slant submanifold of conformal Kenmotsu manifold and $\acute{\nabla}$ (respectively $\acute{\nabla}^\perp$) be induced Riemannian (respectively normal connection) with respect to $\widetilde{\nabla}$ in M (respectively normal bundle $T^\perp M$). Then the Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are given by

$$\widetilde{\nabla}_X Y = \acute{\nabla}_X Y + \acute{h}(X,Y), \tag{38}$$

$$\widetilde{\nabla}_X V = -\acute{A}_V X + \acute{\nabla}^\perp_X V, \tag{39}$$

for any $X, Y \in TM$ and $V \in T^\perp M$. Here $\acute{h}$ and $\acute{A}$ are the second fundamental form and shape operator with respect to $\widetilde{\nabla}$ and are related by

$$g(\acute{h}(X,Y),V) = g(\acute{A}_V X, Y). \tag{40}$$

Lemma 5.3. *Let M be a pseudo-slant submanifold of a conformal Kenmotsu manifold $\widetilde{M}$. Then we have*

$$\acute{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\zeta^T\}, \tag{41}$$

$$\acute{h}(X,Y) = h(X,Y) - \frac{1}{2}g(X,Y)\zeta^\perp, \tag{42}$$

$$\acute{A}_V X = A_V X - \frac{1}{2}\omega(V)X, \tag{43}$$

$$\acute{\nabla}^\perp_X V = \nabla^\perp_X V + \frac{1}{2}\omega(X)V, \tag{44}$$

for any $X, Y \in TM$ *and* $V \in T^{\perp}M$.

Proof. Using Gauss and Weingarten formulas in (5), we get equations from (41) to (44). ■

Corollary 5.4. *Let* M *be a proper pseudo-slant submanifold of conformal Kenmotsu manifold* $\widetilde{M}$. *Then the anti-invariant distribution* $D^{\perp}$ *is integrable if and only if*

$$\acute{A}_{NX}Y = \acute{A}_{NY}X,$$

for any $X, Y \in D^{\perp}$.

Proof. From Theorem 5.1, we know that anti-invariant distribution $D^{\perp}$ of M is integrable if and only if $A_{NY}X - A_{NX}Y = \frac{1}{2}\{\omega(NY)X - \omega(NX)Y\}$. Considering (43) in this we get

$$\acute{A}_{NX}Y = \acute{A}_{NY}X,$$

for any $X, Y \in D^{\perp}$. This completes the proof. ■

Lemma 5.5. *Let* M *be a proper pseudo-slant submanifold of conformal Kenmotsu manifold* $\widetilde{M}$. *Then for any* $X \in D^{\perp}$ *and* $Y \in TM$, *we have*

$$-T(\acute{\nabla}_X Y) = \acute{A}_{NY}X - \omega(NY) - g(X,Y)(T\zeta^T + t\omega^{\perp}) + t\acute{h}(X,Y). \tag{45}$$

Proof. Let $X \in TM$ and $Y \in D^{\perp}$. From (5), (6), (8) and (9), we get

$$\begin{aligned} -A_{NY}X + \nabla^{\perp}_X NY = &-\frac{1}{2}\{\omega(NY)X - \omega(Y)\phi X + g(X,Y)\phi\zeta\} \\ &+ \phi(\nabla_X Y + h(X,Y)). \end{aligned}$$

Again using Lemma 5.3, (11), (12) and (22) in the above equation we get

$$\begin{aligned} -\acute{A}_{NY}X + \acute{\nabla}^{\perp}_X NY = &\frac{1}{2}\omega(NY)X + \omega(Y)TX + \omega(Y)NX \\ &- (T\zeta^T + N\zeta^T + t\zeta^{\perp} + n\zeta^{\perp})g(X,Y) \\ &+ T\acute{\nabla}_X Y + N\acute{\nabla}_X Y + t\acute{h}(X,Y) + n\acute{h}(X,Y). \end{aligned}$$

Thus (45) follows from taking the tangential part of the above equation. ■

Now for any $X \in TM$ and $Y, V \in D^{\perp}$, taking inner product of (45) with by $V \in D^{\perp}$we have

$$\begin{aligned} 0 &= -g(T\acute{\nabla}_X Y, V) \\ &= g(\acute{A}_{NY}V - \acute{A}_{NV}Y, X) - \omega(NY)g(X,V) - g(T\zeta^T + t\zeta^{\perp}, V)g(X,Y). \end{aligned}$$

If $\omega(NY)V + g(T\zeta^T + t\zeta^\perp, V)Y = 0$, then we get $\acute{A}_{NY}V - \acute{A}_{NV}Y = 0$. In view of Corollary 5.4 one can state the following result.

Theorem 5.6. *The anti-invariant distribution $D^\perp$ of proper pseudo-slant submanifold of conformal Kenmotsu manifold $\widetilde{M}$ is integrable if and only if*

$$\omega(NY)V + g(T\zeta^T + t\zeta^\perp, V)Y = 0. \tag{46}$$

for any $Y, V \in D^\perp$.

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