Southeast Asian Bulletin of Mathematics (2023) 47: 869–879

Southeast Asian Bulletin of Mathematics

A Study of the Green's Relations on n-Potent Semirings*

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Received 15 August 2021
Accepted 11 January 2022

Communicated by Xueming Ren

AMS Mathematics Subject Classification(2020): 08B15, 08B05, 16Y60, 20M07

Abstract. In this paper, we study the semirings which satisfy the identities $x^n \approx x$, $(2^n - 1)x \approx x$, $(x + y)^{n-1} \approx x^{n-1} + y^{n-1}$ and $(xy)^{n-1} \approx x^{n-1}y^{n-1}$. We give the characterizations of the binary relations $\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{R}^{+}$ and $\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$, and obtain the sufficient and necessary conditions which make these binary relations be congruences. Finally, we show that the classes of semirings which can be determined by the above congruences are indeed varieties of semirings.

Keywords: Green's relations; Congruences; Semirings; Varieties of semirings.

1. Introduction

By a semiring [13] we mean that an algebra $(S, +, \cdot)$ of type $(2, 2)$ such that

(i) $(S, +)$ is a semigroup;
(ii) $(S, \cdot)$ is a semigroup;
(iii) the distributive laws $x(y + z) \approx xy + xz$ and $(x + y)z \approx xz + yz$ hold in S.

The semigroup $(S, +)$ is called the additive reduct and $(S, \cdot)$ the multiplicative reduct of the semiring $(S, +, \cdot)$. For the sake of simplicity, we use S to denote the system $(S, +, \cdot)$.

*The authors are supported by National Natural Science Foundations of China (Grant No. 11801239, 11971383) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant No. 22JSY023).

A semiring S is called multiplicative idempotent semiring if $(S,\cdot)$ is an idempotent semigroup (band), i.e., if it satisfies the identity $x^2\approx x$. If both $(S,+)$ and $(S,\cdot)$ are idempotent semigroups (bands), i.e., if S satisfies the identities $x^2\approx x$ and $x+x\approx x$, then S is called an idempotent semiring. The variety of all idempotent semirings will be denoted by $\mathbf{I}$ [15, 14]. A semiring S is called n-potent semiring if S satisfies the identity $x^n\approx x$ for n ≥ 2.

By definition, semiring can be regarded as two semigroups on the same nonempty set linked by distributive laws. It is well known that Green's relations play important roles in the development of semigroup theory. Therefore, many scholars use Green's relations [4] of multiplicative semigroups (resp. additive semigroups) on semirings to study semirings. In recent years, some authors have further extended the Green's relations of semigroups to algebraic systems of some given types (see [6] and [13]).

For a semiring S, Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on the multiplicative reduct $(S,\cdot)$ are denoted by $\mathcal{L}^{\cdot}$, $\mathcal{R}^{\cdot}$, $\mathcal{H}^{\cdot}$ and $\mathcal{D}^{\cdot}$, respectively. Dually, Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on the additive reduct $(S,+)$ are denoted by $\mathcal{L}^{+}$, $\mathcal{R}^{+}$, $\mathcal{H}^{+}$ and $\mathcal{D}^{+}$, respectively.

It follows from [5] that

$$\begin{aligned}(a,b)\in\mathcal{L}^{\cdot}&\Longleftrightarrow(\exists u,v\in S^1)\ \ ua=b,vb=a,\\(a,b)\in\mathcal{R}^{\cdot}&\Longleftrightarrow(\exists u,v\in S^1)\ \ au=b,bv=a,\\(a,b)\in\mathcal{L}^{+}&\Longleftrightarrow(\exists s,t\in S^0)\ \ s+a=b,t+b=a,\\(a,b)\in\mathcal{R}^{+}&\Longleftrightarrow(\exists u,v\in S^0)\ \ a+s=b,b+t=a.\end{aligned}$$

It is necessary to note that let ρ_1 and ρ_2 be equivalences. The meet $\rho_1\wedge\rho_2$ means $\rho_1\cap\rho_2$ and the join $\rho_1\vee\rho_2$ means the smallest equivalence generated by ρ_1 and ρ_2.

Many scholars studied idempotent semirings by the Green's relations of multiplicative semigroups (resp. additive semigroups) on semirings. Pastijn and Zhao [7] gave various characterizations for the idempotent semirings whose Green's $\mathscr{D}$-relation on the multiplicative reduct is the least lattice congruence. Zhao et al. [15] studied some subvarieties of $\mathbf{I}$ which are related to Green's $\mathscr{L}$-relation, and provided equational bases for them and conditions guaranteeing that some multiplicative Green's relations are semiring congruences. Zhao et al. [14] studied the classes of idempotent semirings which are related to the relations $\mathcal{D}^{+}$, $\mathcal{D}^{\cdot}$, $\mathcal{D}^{+}\cap\mathcal{D}^{\cdot}$, and $\mathcal{D}^{+}\vee\mathcal{D}^{\cdot}$. They proved that $\mathcal{D}^{\cdot}$ is the congruence on an idempotent semiring if and only if both $\mathcal{D}^{+}\cap\mathcal{D}^{\cdot}$ and $\mathcal{D}^{+}\vee\mathcal{D}^{\cdot}$ are congruences. Pastijn and Zhao [8] studied idempotent semirings with commutative addition, and provided sufficient conditions for $\mathcal{D}^{\cdot},\mathcal{L}^{\cdot}$ and $\mathcal{R}^{\cdot}$ to be congruences. Damljanović et al. [3] obtained the congruence openings of Green's relations on the additive reduct of a semiring and studied the variety of additively idempotent semirings. Sen et al. [11] studied the semiring whose additive reduct is a semilattice and defined two binary relations $\bar{\mathcal{L}}$ and $\bar{\mathcal{R}}$ on a k-regular semiring $(S,+,\cdot)$ analogous to the Green's relations on a regular semigroup. By definition of $\bar{\mathcal{L}}$ and $\bar{\mathcal{R}}$, Basic properties of k-regular semirings whose k-idempotents are commutative have been studied. In 2016, Ren et al. [10] studied the Burnside ai-semiring which satisfies the identity $x^n \approx x$ and characterized Green's relations of the multiplicative reduct on these semirings. The sufficient and necessary conditions for these Green's relations to be congruences are given. In 2020, Cheng and Shao [2] studied several varieties of semirings by means of congruence openings of multiplicative Green's relations on a multiplicatively idempotent semiring.

Let S be a semiring and satisfy the following identities for n ≥ 2:

$$x^n \approx x, \tag{1}$$

$$(2^n - 1)x \approx x, \tag{2}$$

$$(x+y)^{n-1} \approx x^{n-1} + y^{n-1}, \tag{3}$$

$$(xy)^{n-1} \approx x^{n-1}y^{n-1}. \tag{4}$$

Then $(S, \cdot)$ is a completely regular semigroup [9] by $aa^{n-2}a = a$ and $aa^{n-2} = a^{n-1} = a^{n-2}a$ for all $a \in S$. $E_{\cdot}(S)$ is the set of all idempotents of $(S, \cdot)$. By (1), $E_{\cdot}(S) = \{a^{n-1} | a \in S\}$ and by (4), $E_{\cdot}(S)$ is subsemigroup of $(S, \cdot)$, so $(S, \cdot)$ is an orthogroup. By (2), it is trivial to show that $(S, +)$ is a completely regular semigroup and $E_+(S) = \{(2^n - 2)a | a \in S\}$. Let us denote by $\mathbf{COS_n^{\cdot}}$ the class of semirings defined by the additional identities (1), (2), (3) and (4). From Birkhoff's theorem [1], $\mathbf{COS_n^{\cdot}}$ is a semiring variety. It is easy to prove that $\mathbf{I}$ is a subvariety of $\mathbf{COS_n^{\cdot}}$.

In this paper, we mainly study some equivalences which are related to Green's $\mathcal{L}$ relation on any semiring in $\mathbf{COS_n^{\cdot}}$. In Sect. 2 we study some equivalence relations which are the meet of Green's $\mathcal{L}$ relation and other Green's relations on the member of $\mathbf{COS_n^{\cdot}}$ and give the sufficient and necessary conditions which make these equivalence relations be congruences. Moreover, we prove that several classes of semirings defined by the above congruences are some kinds of varieties of semirings.

For $S \in \mathbf{COS_n^{\cdot}}$, by [5] and [9], $\mathcal{H}^+$ is a congruence of the additive reduct $(S, +)$ of S. Moreover, every $\mathcal{H}^+$- class is a maximal subgroup of $(S, +)$. We denote by $\mathcal{H}_a^+$ the $\mathcal{H}^+$-class containing a for any $a \in S$.

Theorem 1.1. *Let S be a semiring in $\mathbf{COS_n^{\cdot}}$. Then, the following results hold:*

(i) *S satisfies the identity $3x \approx x$.*

(ii) *$(S, +)$ is an orthogroup and $(E_+(S), +, \cdot)$ is a subsemiring of S.*

Proof. Let S be a semiring in $\mathbf{COS_n^{\cdot}}$.

(i) By (1) and (3) we have $(x+x)^{n-1} \approx x^{n-1} + x^{n-1}$ and $x^{n-1}(x+x) \approx x+x$. Further, we can easily deduce that

$$\begin{aligned} 4x \approx & 2x + 2x \\ \approx & x^{n-1}(x+x) + x^{n-1}(x+x) \\ \approx & (x^{n-1} + x^{n-1})(x+x) \\ \approx & 2x. \end{aligned}$$

Hence, we have proved that $E_+(S) = \{2a|a \in S\}$, which implies that for any $a \in S$, $2a = (2^n - 2)a$. By adding a to both sides, it follows that $3a = (2^n - 1)a = a$. Therefore, $3x \approx x$ holds on S.

(ii) It can be easily verified that $(S, +)$ is a completely regular semigroup and $E_+(S) = \{2a|a \in S\}$. If $a, b \in S$, then

$$\begin{aligned}(a+b)^{n-1}(2a+2b) =&(a+b)^{n-2}(a+b)(2a+2b)\\ =&(a+b)^{n-2}2(a+b)(a+b)\\ =&2(a+b)^n\\ =&2(a+b),\end{aligned}$$

and

$$\begin{aligned}(a+b)(2a+2b) =&(a+b)(4a+4b)\\ =&(a+b)(2a+2b)+(a+b)(2a+2b)\\ =&(2a+2b)(2a+2b),\\ (a+b)^{n-1}(2a+2b) =&(2a+2b)^n = 2a+2b.\end{aligned}$$

Hence, we can obtain that $2a + 2b = 2(a + b)$, which leads to $(E_+(S), +)$ be a subsemigroup of $(S, +)$. Thus, by summing up the above results, we have shown that $(S, +)$ is an orthogroup. Now, one can easily prove that $(2a)(2b) = 2(ab)$ for any $a, b \in S$. Hence, $(E_+(S), +, \cdot)$ is a subsemiring of S. ∎

Corollary 1.2. *Let S be a semiring in $\mathbf{COS}^{\cdot}_{\mathbf{n}}$. Then every $\mathcal{H}^+$-class is an abelian group.*

Corollary 1.3. *Let S be a semiring in $\mathbf{COS}^{\cdot}_{\mathbf{n}}$ and $(E_+(S), +)$ be a commutative semigroup. Then $(S, +)$ is a commutative Clifford semigroup.*

Proof. If $S \in \mathbf{COS}^{\cdot}_{\mathbf{n}}$ and $(E_+(S), +)$ is a commutative semigroup. This implies that $(S, +)$ is a Clifford semigroup. By [5, Theorem 4.2.1], $(S, +)$ is a strong semilattice of groups. From Theorem 1.1, $(S, +)$ satisfies $3x \approx x$. In other words, $(S, +)$ is a strong semilattice of abelian groups. Then, we have proved that $(S, +)$ is a commutative semigroup. ∎

2. Some Equivalence Classes Related to $\mathcal{L}^{\cdot}$ and $\mathcal{L}^{+}$

In this section, we concentrate on studying the Green's $\mathcal{L}^{\cdot}$ and $\mathcal{L}^{+}$ relations of semirings in $\mathbf{COS}^{\cdot}_{\mathbf{n}}$, and we give some characterizations of $\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{R}^{+}$ and $\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$. Moreover, we obtain the sufficient and necessary conditions which make these binary relations be congruences.

Let $S \in \mathbf{COS}^{\cdot}_{\mathbf{n}}$. By the application of the well known result given in [5]. we can easily characterize $\mathcal{L}^{+}$ and $\mathcal{L}^{\cdot}$ relations as follows:

$$(\forall a, b \in S)\ a\,\mathcal{L}^+\,b \Leftrightarrow a + 2b = a, b + 2a = b,$$
$$(\forall a, b \in S)\ a\,\mathcal{L}^{\cdot}\,b \Leftrightarrow ab^{n-1} = a, ba^{n-1} = b.$$

According to [15, 9], we directly have the following results.

Lemma 2.1. *Let S be a semiring in $\mathbf{COS}_{\mathbf{n}}^{\cdot}$. Then $\mathcal{D}^+$ is a congruence on S, $\mathcal{L}^+$ and $\mathcal{R}^+$ are congruences on $(S, \cdot)$.*

Proof. Let S be a member in $\mathbf{COS}_{\mathbf{n}}^{\cdot}$. Then, it is trivial to see that $\mathcal{D}^+$ is a congruence on $(S, +)$. Next, we claim that $\mathcal{D}^+$ is a congruence on $(S, \cdot)$.

Suppose that $a, b \in S$ and $a\,\mathcal{D}^+\,b$. Then, we can immediately prove that $a + 2b + 2a = a, b + 2a + 2b = b$. Thus, for any $c \in S$

$$ac + 2bc + 2ac = ac,\ bc + 2ac + 2bc = bc.$$
$$ca + 2cb + 2ca = ca, cb + 2ca + 2cb = cb.$$

Therefore, $\mathcal{D}^+$ is indeed a congruence on $(S, \cdot)$, and so $\mathcal{D}^+$ is a congruence on S. Similarly, $\mathcal{L}^+$ and $\mathcal{R}^+$ are congruences on $(S, \cdot)$. ∎

Lemma 2.2. *Let S be a semiring in $\mathbf{COS}_{\mathbf{n}}^{\cdot}$ and $a, b \in S$ with $a\,\mathcal{L}^{\cdot}\,b$. Then $(a + 2b)\,\mathcal{L}^{\cdot}\,(b + 2a)$.*

Now, with the above lemmas, we are able to give some characterization theorems for n-potent semirings.

Theorem 2.3. *Let S be a semiring in $\mathbf{COS}_{\mathbf{n}}^{\cdot}$ and a, b be any elements in S. Then, the following statements hold:*

(i) $a(\mathcal{L}^{\cdot} \wedge \mathcal{D}^+)b$ *if and only if*

$$(\exists u, v \in S)\ a = uv^{n-1}u^{n-1} + 2vu^{n-1}, b = vu^{n-1} + 2uv^{n-1}u^{n-1};$$

(ii) $a(\mathcal{L}^{\cdot} \wedge \mathcal{L}^+)b$ *if and only if*

$$(\exists u, v \in S)\ a = vu^{n-1} + 2uv^{n-1}u^{n-1} + 2vu^{n-1}, b = uv^{n-1}u^{n-1} + 2vu^{n-1};$$

(iii) $a(\mathcal{L}^{\cdot} \wedge \mathcal{R}^+)b$ *if and only if*

$$(\exists u, v \in S)\ a = 2vu^{n-1} + 2uv^{n-1}u^{n-1} + vu^{n-1}, b = 2vu^{n-1} + uv^{n-1}u^{n-1};$$

(iv) $a(\mathcal{L}^+ \wedge \mathcal{D}^{\cdot})b$ *if and only if*

$$(\exists u, v \in S)\ a = uv^{n-1} + 2u + 2v + 2u, b = vu^{n-1} + 2u + 2v + 2u.$$

Proof. (i) Suppose that $a(\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+})b$, by letting $u = a + 2b, v = b + 2a$. Then, we deduce that

$$\begin{aligned}
&uv^{n-1}u^{n-1} + 2vu^{n-1} \\
=&(a+2b)(b+2a)^{n-1}(a+2b)^{n-1} + 2(b+2a)(a+2b)^{n-1} \\
=&(a+2b) + 2(b+2a) \\
=&a + 2b + 2b + 2a \\
=&a, \\
&vu^{n-1} + 2uv^{n-1}u^{n-1} \\
=&(b+2a)(a+2b)^{n-1} + 2(a+2b)(b+2a)^{n-1}(a+2b)^{n-1} \\
=&(b+2a) + 2(a+2b) \\
=&b + 2a + 2a + 2b \\
=&b.
\end{aligned}$$

Conversely, let u, v be elements of S and suppose that $a = uv^{n-1}u^{n-1} + 2vu^{n-1}, b = vu^{n-1} + 2uv^{n-1}u^{n-1}$. Then

$$\begin{aligned}
ab^{n-1} &= uv^{n-1}u^{n-1} + 2uv^{n-1}u^{n-1} + 2vu^{n-1} + 2vu^{n-1} \\
&= uv^{n-1}u^{n-1} + 2vu^{n-1} \\
&= a, \\
ba^{n-1} &= vu^{n-1} + 2vu^{n-1} + 2uv^{n-1}u^{n-1} + 2uv^{n-1}u^{n-1} \\
&= vu^{n-1} + 2uv^{n-1}u^{n-1} \\
&= b.
\end{aligned}$$

Thus, we have proved that $a \, \mathcal{L}^{\cdot} \, b$. Moreover, we have

$$\begin{aligned}
a + 2b + 2a &= (uv^{n-1}u^{n-1} + 2vu^{n-1}) + 2(uv^{n-1}u^{n-1}) + 2vu^{n-1} \\
&= (uv^{n-1}u^{n-1} + 2vu^{n-1}) + 2((uv^{n-1}u^{n-1} + 2vu^{n-1})) \\
&= a, \\
b + 2a + 2b &= vu^{n-1} + 2uv^{n-1}u^{n-1} + 2(vu^{n-1}) + 2uv^{n-1}u^{n-1} \\
&= vu^{n-1} + 2uv^{n-1}u^{n-1} + 2(vu^{n-1} + 2uv^{n-1}u^{n-1}) \\
&= b,
\end{aligned}$$

which implies that $a \, \mathcal{D}^{+} \, b$. Then $a(\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+})b$.

(ii) Let $a(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})b$. By taking $u = b, v = a$, then

$$vu^{n-1} + 2uv^{n-1}u^{n-1} + 2vu^{n-1} = a + 2b + 2a = a + 2a = a,$$

and

$$uv^{n-1}u^{n-1} + 2vu^{n-1} = b + 2a = b.$$

For the converse part, let $a = vu^{n-1} + 2uv^{n-1}u^{n-1} + 2vu^{n-1}, b = uv^{n-1}u^{n-1} + 2vu^{n-1}$, for $u, v \in S$. Then $a + 2b = a, b + 2a = a$ and $uv^{n-1}u^{n-1} \, \mathcal{L}^{\cdot} \, vu^{n-1}$. Thus $a \, \mathcal{L}^{\cdot} \, b$ and $a \, \mathcal{L}^{+} \, b$. This implies that $a(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})b$.

(iii) This part is similar to (ii).

(iv) If $a(\mathcal{L}^+ \wedge \mathcal{D}^{\cdot})b$, by letting $u = ab^{n-1}, v = ba^{n-1}$, then

$$\begin{aligned} uv^{n-1} + 2u + 2v + 2u &= (u + 2v + 2u)(v + 2u)^{n-1} \\ &= (ab^{n-1})(ba^{n-1})^{n-1} \\ &= ab^{n-1}a^{n-1} = a, \\ vu^{n-1} + 2u + 2v + 2u &= (v + 2u)(u + 2v + 2u)^{n-1} \\ &= (ba + 2ab)(ab + 2ba + 2ab) \\ &= (ba^{n-1})(ab^{n-1})^{n-1} \\ &= ba^{n-1}b^{n-1} = b. \end{aligned}$$

Conversely, let u, v be elements of S and suppose that $a = (u + 2v + 2u)(v + 2u)^{n-1}, b = (v + 2u)(u + 2v + 2u)^{n-1}$. Then $(u + 2v + 2u)\,\mathcal{L}^+(v + 2u)$. By Lemma 2.1, $\mathcal{L}^+$ is a congruence on $(S, \cdot)$, thus $a\,\mathcal{L}^+\,b$.

$$\begin{aligned} ab^{n-1}a^{n-1} =&((u + 2v + 2u)(v + 2u)^{n-1})((u + 2v + 2u)^{n-1}(v + 2u)^{n-1}) \\ =&(u + 2v + 2u)(v + 2u)^{n-1})((u + 2v + 2u)(v + 2u)^{n-1}))^{n-1} \\ =&a, \end{aligned}$$

$$\begin{aligned} ba^{n-1}b^{n-1} =&((v + 2u)(u + 2v + 2u)^{n-1})((v + 2u)^{n-1}(u + 2v + 2u)^{n-1}) \\ =&((v + 2u)(u + 2v + 2u)^{n-1})((v + 2u)(u + 2v + 2u)^{n-1})^{n-1} \\ =&b. \end{aligned}$$

Therefore, $a\,\mathcal{L}^{\cdot}\,b$, and so $a(\mathcal{L}^+ \wedge \mathcal{D}^{\cdot})b$. ■

The above theorem gives some characterizations of $\mathcal{L}^{\cdot} \wedge \mathcal{D}^+$, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^+$, $\mathcal{L}^{\cdot} \wedge \mathcal{R}^+$ and $\mathcal{L}^+ \wedge \mathcal{D}^{\cdot}$. However, these binary relations are not congruences in general. Here are two examples as follows.

Example 2.4. Let $(S, +, \cdot)$ be a semiring with the following addition and multiplication tables:

$+$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

$\cdot$	a	b	c	d	e
a	a	d	e	d	e
b	a	b	b	d	e
c	a	c	c	d	e
d	a	d	d	d	e
e	a	e	e	d	e

It is routine to check that $(S, \cdot)$ is a band, and $(S, +)$ is a left zero band with $\mathcal{L}^+ = \mathcal{D}^+ = \nabla$. So we can easily see that S is a semiring in $\mathbf{COS_2^{\cdot}}$. Further, from the multiplication table, we can check $cb = c, bc = b$, which implies that

$c\,\mathcal{L}^{\cdot}\,b$. So we have $c(\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+})b$ and $c(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})b$. But $(ac, ab) \notin \mathcal{L}^{\cdot}$. Thus $\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+}$ and $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ are not congruences on S.

Example 2.5. Let $(S, +, \cdot)$ be a semiring with the following addition and multiplication tables:

$+$	a	b	c	d	e
a	a	d	e	d	e
b	a	b	b	d	e
c	a	c	c	d	e
d	a	d	d	d	e
e	a	e	e	d	e

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

It can be easily checked that $(S, +)$ is a band, and $(S, \cdot)$ is a left zero band with $\mathcal{L}^{\cdot} = \mathcal{D}^{\cdot} = \nabla$. Further, S is a semiring in $\mathbf{COS}^{\cdot}_{\mathbf{2}}$. From the addition table, $b + 2c = b + c = b, c + 2b = c + b = c$. Thus, we have $c\,\mathcal{L}^{+}\,b$ and also have $c(\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot})b$. But $(a+b)+2(a+c) = d+e = e \neq a+b, (a+c)+2(a+b) \neq a+c$, that is $(a+b, a+c) \notin \mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$. So $\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$ is not a congruence on S.

In the following, we shall give the sufficient and necessary conditions which make $\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$, $\mathcal{L}^{\cdot} \wedge \mathcal{R}^{+}$ and $\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$ be congruences.

Theorem 2.6. *The following statements hold for a semiring S in $\mathbf{COS}^{\cdot}_{\mathbf{n}}$:*

(i) $\mathcal{L}^{\cdot} \wedge \mathcal{D}^{+}$ *is a congruence on S if and only if S satisfies the following identities:*

$$\begin{aligned}(s(x,y)+z)(t(x,y)+z)^{n-1} &\approx s(x,y)+z,\\ (z+s(x,y))(z+t(x,y))^{n-1} &\approx z+s(x,y),\\ (zs(x,y))(zt(x,y))^{n-1} &\approx zs(x,y),\end{aligned}$$

where $s(x,y) = yx^{n-1} + 2xy^{n-1}x^{n-1}, t(x,y) = xy^{n-1}x^{n-1} + 2yx^{n-1}$.

(ii) $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ *is a congruence on S if and only if S satisfies the following identities:*

$$\begin{aligned}(z+p(x,y))+2(z+q(x,y)) &\approx z+p(x,y),\\ (z+p(x,y))(z+q(x,y))^{n-1} &\approx z+p(x,y),\\ (p(x,y)+z)+2(q(x,y)+z) &\approx p(x,y)+z,\\ (zp(x,y))(zq(x,y))^{n-1} &\approx zp(x,y),\end{aligned}$$

where $p(x,y) = xy^{n-1}x^{n-1} + 2yx^{n-1}, q(x,y) = yx^{n-1} + 2xy^{n-1}x^{n-1} + 2yx^{n-1}$.

(iii) $\mathcal{L}^{\cdot} \wedge \mathcal{R}^{+}$ *is a congruence on S if and only if S satisfies the following iden-*

tities:

$$\begin{aligned} 2(m(x,y)+z)+(w(x,y)+z) &\approx w(x,y)+z, \\ (w(x,y)+z)(m(x,y)+z)^{n-1} &\approx w(x,y)+z, \\ (z+w(x,y))(z+m(x,y))^{n-1} &\approx z+w(x,y), \\ (zw(x,y))(zm(x,y))^{n-1} &\approx zw(x,y), \end{aligned}$$

where $w(x,y) = 2yx^{n-1} + 2xy^{n-1}x^{n-1} + yx^{n-1}, m(x,y) = 2yx^{n-1} + xy^{n-1}x^{n-1}$.

(iv) $\mathcal{L}^{+} \wedge \mathcal{D}^{\cdot}$ *is a congruence on* S *if and only if* S *satisfies the following identities:*

$$\begin{aligned} (z+u(x,y))+2(z+v(x,y)) &\approx z+u(x,y), \\ (z+u(x,y))(z+v(x,y))^{n-1}(z+u(x,y)^{n-1} &\approx z+u(x,y), \\ (u(x,y)+z)(v(x,y)+z)^{n-1}(u(x,y+z)^{n-1} &\approx u(x,y)+z, \end{aligned}$$

where $u(x,y) = xy^{n-1}+2x+2y+2x, v(x,y) = yx^{n-1}+2x+2y+2x.$

Proof. We only prove (ii). (i),(iii) and (iv) can be proved similarly.

Let S be a semiring in $\mathbf{COS_n^{\cdot}}$ and $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ be a congruence on S. Then by (ii) in Theorem 2.3, for any $u, v \in S$, $p(u,v)(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})q(u,v)$ in which $p(u,v) = uv^{n-1}u^{n-1}+2vu^{n-1}$ and $q(u,v) = vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}$. Consequently, for $w \in S$

$$\begin{aligned} (vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}+w) &\ (\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})\ (uv^{n-1}u^{n-1}+2vu^{n-1}+w), \\ (w+vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}) &\ (\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})\ (w+uv^{n-1}u^{n-1}+2vu^{n-1}+2vu), \\ w(vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}) &\ (\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})\ w(uv^{n-1}u^{n-1}+2vu^{n-1}). \end{aligned}$$

These imply that

$$\begin{aligned} (w+vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}) &\ \mathcal{L}^{+}\ (w+uv^{n-1}u^{n-1}+2vu^{n-1}), \\ (w+vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}) &\ \mathcal{L}^{\cdot}\ (w+uv^{n-1}u^{n-1}+2vu^{n-1}), \\ (vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}+w) &\ \mathcal{L}^{\cdot}\ (uv^{n-1}u^{n-1}+2vu^{n-1}+w), \\ w(vu^{n-1}+2uv^{n-1}u^{n-1}+2vu^{n-1}) &\ \mathcal{L}^{\cdot}\ w(uv^{n-1}u^{n-1}+2vu^{n-1}). \end{aligned}$$

Thus, by (4),(5)

$$\begin{aligned} (w+p(u,v))+2(w+q(u,v)) &\approx w+p(u,v), \\ (w+p(u,v))(w+q(u,v))^{n-1} &\approx w+p(u,v), \\ (p(u,v)+w)+2(q(u,v)+w) &\approx p(u,v)+w, \\ (wp(u,v))(wq(u,v))^{n-1} &\approx wp(u,v). \end{aligned}$$

Thereby S satisfies these identities.

Conversely, we suppose that S is a semiring in $\mathbf{COS_n^{\cdot}}$ satisfying these identities, and $a(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})b$ for some $a, b \in S$. It follows from Theorem 2.3, there exist

$u, v \in S$ such that $a = uv^{n-1}u^{n-1} + 2vu^{n-1}, b = vu^{n-1} + 2uv^{n-1}u^{n-1}$. For any $c \in S$, from these identities

$$(c+b) + 2(c+a) = c+b, (c+b)(c+a)^{n-1} = (c+b).$$

By interchanging a and b above,

$$(c+a) + 2(c+b) = c+a, (c+a)(c+b)^{n-1} = (c+a).$$

That is $(c+a)(\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+})(c+b)$. Consequently, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ is a left congruence on the additive reduct $(S, +)$. From these identities, $(b+c)(a+c)^{n-1} = (b+c)$. By interchanging a and b again, $(a+c)(b+c)^{n-1} = (a+c)$. Thus, $(a+c)\,\mathcal{L}^{\cdot}(b+c)$. Now, by the above two formulas, we get

$$(a+c)(\dot{\mathcal{L}} \wedge \overset{+}{\mathcal{L}})(b+c).$$

This shows that $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ is a right congruence $(S, +)$. For the multiplicative reduct $(S, \cdot)$, by Lemma 2.1 that $\mathcal{L}^{+}$ is a congruence on $(S, \cdot)$ and $\mathcal{L}^{\cdot}$ is right congruence on $(S, \cdot)$, so $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ is a right congruence on $(S, \cdot)$. Finally, we prove that $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ is also a left congruence on $(S, \cdot)$. From these identities we have $(cb)(ca)^{n-1} = cb$. And by interchanging a and b, we can similarly obtain

$$(ca)(cb)^{n-1} = ca.$$

In conclusion, we have $cb\,\mathcal{L}^{\cdot}\,ca$, and so $(ca)\,\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}(cb)$. Thus, $\mathcal{L}^{\cdot} \wedge \mathcal{L}^{+}$ is a left congruence on $(S, +, \cdot)$. ■

We define some subclasses of $\mathbf{COS_n^{\cdot}}$ in the following way

$$\{S \in \mathbf{COS_n^{\cdot}} : \mathcal{L}^{\cdot} \wedge \mathcal{D}^{+} \in Con(S)\}, \{S \in \mathbf{COS_n^{\cdot}} : \mathcal{L}^{\cdot} \wedge \mathcal{L}^{+} \in Con(S)\},$$
$$\{S \in \mathbf{COS_n^{\cdot}} : \mathcal{L}^{\cdot} \wedge \mathcal{R}^{+} \in Con(S)\}, \{S \in \mathbf{COS_n^{\cdot}} : \mathcal{L}^{+} \wedge \mathcal{D}^{\cdot} \in Con(S)\}.$$

By Theorem 2.6, these semirings classes are subvarieties of $\mathbf{COS_n^{\cdot}}$.

Since $\mathbf{I}$ is subvariety of $\mathbf{COS_n^{\cdot}}$, Theorems 2.3 and 2.6 also hold in $\mathbf{I}$. Thus some results obtained by Zhao, Shum and Guo [15] are generalized and extended.

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