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The Conjugacy Classes Ranks of the Alternating Simple Group A_{11}*

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Abstract. Let G be a finite group and X be a conjugacy class of G. The *rank* of X in G, denoted by $rank(G : X)$, is defined to be the minimum number of elements of X generating G. We investigate the ranks of the alternating group A_{11}. We use the structure constants method to determine the ranks of all the non-trivial classes of the group A_{11}.

Keywords: Conjugacy classes; Rank; Generation; Alternating simple group.

1. Introduction

Let G be a finite group and nX a non-identity conjugacy class of G. We define $rank(G : nX)$ to be the minimum number of elements of G in nX that generate G. This is called the rank of nX in G.

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One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see [16]). Moori in various papers (see [9], [10] and [11]), computed the ranks of the involuntary classes of the Fischer sporadic simple group Fi_{22} and his results were that $rank(Fi_{22} : 2A) \in \{5, 6\}$ and $rank(Fi_{22} : 2B) = 3 = rank(Fi_{22} : 2C)$. On the other hand, the work of Hall and Soicher [8] implies that $rank(Fi_{22} : 2A) = 6$.

In this paper, we determine the rank for each non-identity conjugacy class of the group A_{11}. We follow some of the methods used in the paper written by Basheer and Moori [2], and the techniques used by Ganief when he computed (p, q, r)-generations of certain groups [4].

2. Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant.* With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \qquad (1)$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle. \qquad (2)$$

Definition 2.1. *If* $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, *the group* G *is said to be* $(C_1, C_2, \cdots, C_k)$*-generated.*

Remark 2.2. A group G is $(C_1, C_2, \cdots, C_k)$-generated if and only if $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$.

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \cdots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \cdots, h_{k-1})$ such that

$$h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \cdots, h_{k-1} \rangle \leq H. \qquad (3)$$

The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.3. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [5] and [6, Theorem 2.1]. ∎

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \cdots, C_k)$, namely $\Delta_G^*(C_1, C_2, \cdots, C_k)$, where

$$\Delta_G^*(C_1, \cdots, C_k) \geq \Delta_G(C_1, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \cdots, C_k), \quad (4)$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \cdots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups except for $G = \mathbb{M}$ the Monster group, it is possible to build a small subroutine in GAP [14] to compute the values of $\Delta_G^* = \Delta_G(C_1, C_2, \cdots, C_k)$ for any collection of conjugacy classes and for any alternating simple group.

The following results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.4. *Let G be a finite centerless group. If $\Delta_G^*(C_1, C_2, \cdots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta_G^*(C_1, C_2, \cdots, C_k) = 0$ and therefore G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. See [2, Lemma 2.7]. ∎

Theorem 2.5. [12] *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

By the Atlas of finite group representations [15], the alternating group A_{11} is acting on 11 points, so that $n = 11$. Since our generation is triangular, we have $s = 3$. Hence if A_{11} is (l, m, n)-generated, then $\sum c_i \leq 13$.

Theorem 2.6. [13] *Let $g_1, g_2, \cdots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with* $\dim \mathbb{V} = n \geq 2$. *Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([4]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi{\downarrow}^G_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{5}$$

The following results are in some cases useful in determining the ranks finite groups.

Theorem 2.7. [2, Lemma 2.5] *Let G be a $(2X, sY, tZ)$-generated simple group. Then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 2.8. [1] *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $((\underbrace{lX, lX, \ldots, lX}_{m-times}), (nZ)^m)$-generated.*

Corollary 2.9. [1] *Let G be a finite simple group such that G is $(lX, mYnZ)$-generated. Then $rank(G : lX) \leq m$.*

Proof. The result follows immediately from Lemma 2.8. ■

Theorem 2.10. [7] *Let G be a $(2X, sY, tZ)$-generated simple group. Then G is $(sY, sY, (tZ)^2)$-generated.*

Corollary 2.11. *Let G be a finite simple group such that G is $(2X, mY, nZ)$-generated. Then $rank(G : mY) = 2$.*

Proof. Since G is (lX, mY, nZ)-generated so by Lemma 2.8 we obtained that G is $(mY, mY, (nZ)^m)$-generated. Hence the result follows. ■

3. The Alternating Group A_{11}

In this section we apply the results discussed in Section 2, to the alternating group A_{11}. We determine the ranks for all the nonidentity conjugacy classes of A_{11}. The alternating group A_{11} is a simple and has order $19958400 = 2^7 \times 3^4 \times 5^2 \times 7 \times 11$. From [3], the group G has exactly 31 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$H_1 = A_{10}$ $\quad$ $H_2 = S_9$ $\quad$ $H_3 = (A_8 \times 3){:}2$
$H_4 = (A_7 \times A_4){:}2$ $\quad$ $H_5 = (A_6 \times A_5){:}2$ $\quad$ $H_6 = M_{11}$
$H_7 = M_{11}$.

Throughout this paper, by G we always mean the alternating group A_{11}, unless stated otherwise. It is well-known that G can be generated in terms of permutations on 11 points. From GAP or the electronic Atlas of finite group representations [15], the following two elements g_1 and g_2 generate G where:

$$g_1 = (1, 2, 3)$$
$$g_2 = (3, 4, 5, 6, 7, 8, 9, 10, 11),$$

with $o(g_1) = 3,\ o(g_2) = 9$ and $o(g_1g_2) = 11$.

In Table 1 we list representatives of classes of the maximal subgroups together with the orbits lengths of G on these groups and the permutation characters.

In Table 2, we list the values of the cyclic structure for each conjugacy of G which containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 3 gives the partial fusion maps of classes of maximal subgroups into the classes of G. These will be used in our computations.

Table 1: Maximal subgroups of G

Maximal Subgroup	Order	Orbit Lengths	Character
H_1	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	[1,10]	$1a + 10a$
H_2	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	[2,9]	$1a + 10a + 44a$
H_3	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	[3,8]	$1a + 10a + 44a + 110a$
H_4	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	[7,4]	$1a + 10a + 44a + 110a + 165a$
H_5	$2^6 \cdot 3^3 \cdot 5^2$	[5,6]	$1a + 10a + 44a + 110a$ +132a + 165a
H_6	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$
H_7	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$

4. The Conjugacy Class Ranks of G

Now we study the ranks of G with respect to the various conjugacy classes of all its nonidentity elements. We start our investigation on the ranks of the non-trivial classes of G by looking at the two classes of involutions $2A$ and $2B$. It is well known that the rank of any of these involutions classes will be at least 3.

Table 2: Cycle structures of conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$2A$	$1^7 2^2$	9	2
$2B$	$1^3 2^4$	7	4
$3A$	$1^8 3^1$	9	2
$3B$	$1^5 3^2$	7	4
$3C$	$1^2 3^3$	5	6
$4A$	$1^5 2\ 3$	7	4
$4B$	$1^2 4^2$	5	6
$4C$	$1^1 2^3 4^1$	5	6
$5A$	$1^6 5^1$	7	4
$5B$	$1^1 5^2$	3	8
$6A$	$1^4 3^1$	5	6
$6B$	$1^4 2^2 3^1$	7	4
$6C$	$1^2 2^2 3^2$	5	6
$6D$	$1^3 2^1 6^1$	5	6
$6E$	$2^1 3^1 6^1$	3	8
$7A$	$1^4 7^1$	5	6
$8A$	$1^2 2^1 8^1$	3	8
$9A$	$1^2 9^1$	3	8
10	$1^2 2^2 5^1$	5	6
$11A$	11^1	1	10
$11B$	11^1	1	10
$12A$	$3^1 4^2$	3	8
$12B$	$1^2 2^1 3^1 4^1$	5	6
$12C$	$1^1 4^1 6^1$	3	8
$14A$	$2^2 7^1$	3	8
$15A$	$1^3 3^1 5^1$	5	6
$15B$	$3^2 5^1$	5	6
20	$2^1 4^1 5^1$	3	8
$21A$	$1^1 3^1 5^1$	3	8
$21B$	$1^1 3^1 5^1$	3	8

The group G is not $(2Y, 2Z, pX)$-generated, for if G is $(2Y, 2Z, pX)$-generated, then G is a dihedral group and thus is not simple for all $Y, Z \in \{A, B\}$. Also we know that if G is (l, m, n)-generated with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ and G is simple, then $G \cong A_5$, but $G \cong A_{11}$ and $A_{11} \ncong A_5$.

Lemma 4.1. $rank(G : 2A) \notin \{3, 4\}$.

Proof. Now if G is $(2A, 2A, 2A, nX)$-generated, then by Scott's Theorem [13] we must have $d_{2A} + d_{2A} + d_{2A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 2 that $3 \times d_{2A} + d_{nX} = 3 \times 2 + d_{nX} < 20$ for each nX, where nX is a set of all the non-

Table 3: The partial fusion maps into G

H_1-class	2a	2b	3a	3b	3c	5a	5b	7a						
$\to G$	2A	2B	3A	3B	3C	5A	5B	7A						
h						6	1	4						
H_2-class	2a	2b	2c	2d	3a	3b	3c	5a	7a					
$\to G$	2A	2A	2B	2B	3C	3A	3B	5A	7A					
h								15	6					
H_3-class	2a	2b	2c	2d	3a	3b	3c	3d	3e	5a	7a			
$\to G$	2B	2B	2A	2A	3A	3B	3C	3A	3B	5A	7A			
h										20	4			
H_4-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	7a		
$\to G$	2A	2A	2A	2B	2B	3A	3A	3B	3B	3C	5A	7A		
h											15	1		
H_5-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	5b	5c	5d
$\to G$	2A	2A	2B	2A	2B	3A	3B	3A	3B	3C	5A	5A	5B	5B
h											1	6	1	1
H_6-class	2a	3a	5a	11a	11b									
$\to G$	2B	3C	5B	11A	11B									
h			5	1	1									
H_7-class	2a	3a	5a	11a	11b									
$\to G$	2B	3C	5B	11A	11B									
h			5	1	1									

identity classes of G and therefore G is not $(2A, 2A, 2A, nX)$-generated, for any nX. We use the similar arguments to prove that G is not $(2A, 2A, 2A, 2A, nX)$-generated because $4 \times d_{2A} + d_{nX} = 4 \times 2 + d_{nX} < 20$ for any $nX \in T$. Hence $rank(G : 2A) \notin \{3, 4\}$. ■

Proposition 4.2. $rank(G : 2A) = 5$.

Proof. From Table 3 we see that H_6 (or H_7) (two non-conjugate copies) is the only maximal subgroup containing elements of orders 2, 5 and 11. The intersection of H_6 from one conjugacy class with H_7 from a different conjugacy class has no element of order 11. No element of order 2 from these two maximal subgroups fuses to the class $2A$ of G. We then obtained that $\Delta^*_G(2A, 5B, 11X) = \Delta_G(2A, 5B, 11X) = 44 > 11 = |C_G(11X)|$ for $X \in \{A, B\}$. This proves that G is $(2A, 5B, 11X)$-generated for $X \in \{A, B\}$. Since G is $(2A, 5B, 11X)$-generated for $X \in \{A, B\}$, by Corollary 2.9, we must have $rank(G : 2A) \leq 5$. Since by Lemma 4.1, $rank(G : 2A) \notin \{3, 4\}$, it follows that $rank(G : 2A) = 5$. ■

Lemma 4.3. *The group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$.*

Proof. From Table 3 we see that H_6 (or H_7) (two non-conjugate copies) is the only maximal subgroup containing elements of orders 2, 3 and 11. We obtained that $\sum_{H_6}(2a, 3a, 11x) = 11$ and $h(11X, H_6) = 1$ (or $\sum_{H_7}(2a, 3a, 11x) = 11$ and $h(11X, H_7) = 1$). We obtained that $\Delta_G(2B, 3C, 11X) = 110$ and it follows that $\Delta^*_G(2B, 5B, 11X) = \Delta_G(2B, 3C, 11X) - \sum_{H_6}(2a, 3a, 11x) - \sum_{H_7}(2a, 3a, 11x) = 110 - 11 - 11 = 88 > 11 = |C_G(11X)|$ for $X \in \{A, B\}$. This proves that G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$. ■

Proposition 4.4. $rank(G : 2B) = 3$.

Proof. Since by Lemma 4.3, the group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$, by Corollary 2.9, we must have $rank(G : 2B) \leq 3$. It then follows that $rank(G : 2B) = 3$. ■

Proposition 4.5. $rank(G : 3A) = 5$.

Proof. Now if G is $(3A, 3A, nX)$-generated, then by Scott's Theorem [13] we must have $d_{3A} + d_{3A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 2 that $2 \times d_{3A} + d_{nX} = 2 \times 2 + d_{nX} < 20$ for each non-identity class of G and therefore G is not $(3A, 3A, nX)$-generated. We use similar arguments to prove that G is not $(3A, 3A, 3A, nX)$- and $(3A, 3A, 3A, 3A, nX)$-generated because we obtained that $3 \times d_{2A} + d_{nX} = 3 \times 2 + d_{nX} < 20$ and $4 \times d_{2A} + d_{nX} = 4 \times 2 + d_{nX} < 20$ for any non-identity nX of G.

By Table 3 we see that no maximal subgroup of G meets the classes $3A$, $5B$ and $11A$ of G. We then obtained that $\Delta^*_G(3A, 5B, 11A) = \Delta_G(3A, 5B, 11A) = 11 > 0$, proving that G is $(3A, 5B, 11A)$-generated group. By applying Lemma 2.8, it follows that G is $(3A, 3A, 3A, 3A, 3A, (11A)^5)$-generated. Using GAP, $(11A)^5 = 11A$ so that G becomes $(3A, 3A, 3A, 3A, 3A, 11A)$-generated. Since $rank(G : 3A) \notin \{2, 3, 4\}$, it follows that $rank(G : 3A) = 5$. ■

Proposition 4.6. $rank(G : 3B) = 3$.

Proof. If the group G is $(3B, 3B, nX)$-generated then we must have $c_{3B} + c_{3B} + nX \leq 13$ where nX is any non-identity class of G. Since by Table 2 we have $c_{3B} + c_{3B} + c_{nX} = 7 + 7 + c_{nX} > 13$, using Ree's Theorem [12], it follows that G is not $(3B, 3B, nX)$-generated. Thus $rank(G : 3B) \notin 2$.

By Table 3 we see that no maximal subgroup of G meets the classes $3B$, $3C$ and $11A$ or $11B$ of G. We then obtained that $\Delta^*_G(3B, 3C, 11X) = \Delta_G(3B, 3C, 11X) = 66 > 0$, proving that G is $(3B, 3C, 11X)$-generated for $X \in \{A, B\}$. By applying Lemma 2.8, then we obatined that the group G is $(3B, 3B, 3B, (11X)^3)$-generated for all $X \in \{A, B\}$. It is easy to check with GAP that $(11A)^3 = 11A$ and $(11B)^3 = 11B$. Thus G becomes $(3B, 3B, 3B, 11X)$-generated for $X \in \{A, B\}$. Hence $rank(G : 3B) = 3$. ■

Proposition 4.7. $rank(G : 3C) = 2$.

Proof. Since by Lemma 4.3, the group G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$, by Corollary 2.11, it follows that $rank(G : 3C) = 2$. ■

Proposition 4.8. $rank(G : 4A) = 3$.

Proof. If the group G is $(4A, 4A, nX)$-generated then we must have $c_{4A} + c_{4A} + c_{nX} \leq 13$ where nX is any non-identity class of G. Since by Table 2 we have $c_{3B} + c_{3B} + c_{nX} = 7 + 7 + c_{nX} > 13$, using Ree's Theorem [12], it follows that G is not $(3B, 3B, nX)$-generated. Thus $rank(G : 4A) \notin 2$.

By Table 3 we see that no maximal subgroup of G meets the classes $3A$, $4A$ and $11A$ of G. We then obtained that $\Delta^*_G(3A, 4A, 11A) = \Delta_G(3A, 4A, 11A) = 132 > 0$, proving that G is $(3A, 4A, 11A)$-generated. By applying Lemma 2.8, then we obatined that the group G is $(4A, 4A, 4A, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(4A, 4A, 4A, 11A)$-generated. Hence $rank(G : 4A) = 3$. ■

Proposition 4.9. $rank(G : 5A) = 3$.

Proof. Now if G is $(5A, 5A, nX)$-generated, then by Scott's Theorem we must have $d_{5A} + d_{5A} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 2 that $2 \times d_{5A} + d_{nX} = 2 \times 4 + d_{nX} < 20$ for each nX a non-identity class of G and therefore G is not $(5A, 5A, nX)$-generated. Thus $rank(G : 5A) \notin 2$.

By Table 3 we see that no maximal subgroup of G meets the classes $3C$, $5A$ and $11A$ of G. We then obtained that $\Delta^*_G(3C, 5A, 11A) = \Delta_G(3C, 5A, 11A) = 22 > 0$, proving that G is $(3C, 5A, 11A)$-generated. Applying Lemma 2.8, we obatin that the group G is $(5A, 5A, 5A, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(5A, 5A, 5A, 11A)$-generated. Hence $rank(G : 5A) = 3$. ■

Proposition 4.10. $rank(G : 6B) = 3$.

Proof. Now if G is $(6B, 6B, nX)$-generated, then by Scott's Theorem we must have $d_{6B} + d_{6B} + d_{nX} \geq 2 \times 10$. However, it is clear from Table 2 that $2 \times d_{6B} + d_{nX} = 2 \times 4 + d_{nX} < 20$ for each nX a non-identity class of G and therefore G is not $(6B, 6B, nX)$-generated. Thus $rank(G : 6B) \notin 2$.

By Table 3 we see that no maximal subgroup of G meets the classes $3C$, $6B$ and $11A$ of G. We obtain that $\Delta^*_G(3C, 6B, 11A) = \Delta_G(3C, 6B, 11A) = 330 > 0$, proving that G is $(3C, 6B, 11A)$-generated. By applying Lemma 2.8, then we obatined that the group G is $(6B, 6B, 6B, (11A)^3)$-generated. Since $(11A)^3 = 11A$, the group G will become $(6B, 6B, 6B, 11A)$-generated. Hence $rank(G : 6B) = 3$. ■

Proposition 4.11. *Let* $nX \in T := \{4B, 4C, 5B, 6A, 6C, 6D, 6E, 7A, 8A, 9A, 10A, 11A, 11B, 12A, 12B, 12C, 14A, 15A, 15B, 20A, 21A, 21B\}$. *Then* $rank(G : nX) = 2$.

Proof. From Table 3 we see that H_6 (or H_7) (two non-conjugate copies) is the only maximal subgroup containing elements of order 11. The intersection of H_6 from one conjugacy class with H_7 from a different conjugacy class has no element of order 11. In Table 4, we listed we list the values of Δ_G, h and Δ^*_G for all $nX \in T$. Since $\Delta^*_G(nX, nX, 11A) > 11 = |C_G(23A)|$, it follows that G is $(nX, nX, 11A)$-generated where $nX \in T$. This proves that $rank(G : nX) = 2$ for all $nX \in T$. ∎

The main result of this paper is summarized by the following theorem.

Theorem 4.12. *For the alternating group G, we have*

(i) $rank(G : 2A) = rank(G : 3A) = 5$,

(ii) $rank(G : 2B) = rank(G : 3B) = rank(G : 4A) = rank(G : 5A) = rank(G : 6B) = 3$,

(iii) $rank(G : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 3A, 3B, 4A, 5A, 6B\}$ *and where* nX *is a conjugacy class of* G.

Proof. (i) See Propositions 4.2 and 4.5.

(ii) The results follow by the proofs of Propositions 4.4, 4.6, 4.8, 4.9 and 4.10.

(iii) See Propositions 4.7 and 4.11. ∎

Table 4 gives the partial structure contants of G computed using GAP together with the relevant information need in the calculations Δ^*_G. We give some information about $\Delta_G(nX, nX, 11A) = \Delta_G$, $h(11A, M_6)$ (or $h(11A, M_7)$), $\sum_{M_6}(nx, nx, 11a) = \sum_{M_6}$ and $\sum_{M_7}(nx, nx, 11a) = \sum_{M_7}$. The last column $\Delta^*_G(nX, nX, 11A) = \Delta^*_G$ establishes each generation of G by its triples $(nX, nX, 11A)$.

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Table 4: Some information on the classes $nX \in T$

nX	Δ_G	h	$h\sum_{M_6}$	$h\sum_{M_7}$	Δ_G^*
$4B$	1320	1	77	77	1166
$4C$	2640	1	0	0	2640
$5B$	31680	1	297	297	31086
$6A$	55	1	0	1	55
$6C$	3960	1	0	0	3960
$6D$	8800	1	0	0	8800
$6E$	55220	1	154	154	54912
$7A$	825	1	0	0	825
$8A$	318780	1	429	429	317922
$9A$	221760	1	0	0	221760
$10A$	11880	1	0	0	11880
$11A$	147600	1	35	35	147530
$11B$	162000	1	80	80	161840
$12A$	80850	1	0	0	80850
$12B$	31680	1	0	0	31680
$12C$	139260	1	0	0	139260
$14A$	23265	1	0	0	23265
$15A$	6160	1	0	0	6160
$15B$	8976	1	0	0	8976
$20A$	44748	1	0	0	44748
$21A$	44880	1	0	0	44880
$21B$	44880	1	0	0	44880

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