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Extended Hyperbolic Function and its Properties

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Abstract. Aim of this paper is to introduce extended hyperbolic function by using a modified extension of beta function [12] and to establish new properties like integral representation, Mellin transform and many more. Furthermore, we apply Prabhakar fractional integral operator, Caputo-Fabrizio operator and Atangana-Baleanu operator on it. Other than this, we present a graphical representation of the extended hyperbolic function with different values of α also a graphical comparison between Caputo-Fabrizio operator and Atangana-Baleanu operator of hyperbolic function for different values of r.

Keywords: Extended beta function; Extended Hyperbolic function; Prabhakar function; Mittag-Leffler function; Caputo-Fabrizio derivative; Atangana-Baleanu derivative.

1. Introduction

The Mittag Leffler function is a special function playing a key role in the solution of fractional order differ-integral equation. It is also used in various fields of science and engineering, eg. Random walk, Levy flights, fluid flow, etc. (see [7, 8, 9]). In one parameter, the classical Mittag-Leffler function is written as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

here z is a complex variable, $R(\alpha) \geq 0$. If $\alpha = 1$, function is exponential and if $0 < \alpha < 1$, it exists between exponential and hypergeometric function $\frac{1}{1-z}$. The expression of (1), with two parameter α, β has the following form where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{2}$$

$\alpha, \beta \in \mathbb{C}, (Re(\alpha) > 0, Re(\beta) > 0)$.

Prabhakar introduced generalized Mittag-Leffler in [10], $E^{\lambda}_{\alpha,\beta}$ defined as below

$$E^{\lambda}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n z^n}{\Gamma(\alpha n + \beta) n!}, \tag{3}$$

where $\alpha, \beta, \lambda \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\lambda) > 0$. And $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ denotes a pochhammer symbol.

In this paper, we have involved some newly introduced fractional differential and integral operators, defined as follows:

Prabhakar Fractional Integral

With the help of equation (3), the Prabhakar kernel was given by

$$e^{\gamma}_{\alpha,\beta}(w;t) = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(wt^{\alpha}), \tag{4}$$

where $t \in R$, $\alpha, \beta, \gamma, w \in \mathbb{C}$, $Re(\alpha) > 0$. With the help of the above function Prabhakar introduced Prabhakar fractional integral (see [5],[8], [6]) as:

$$E^{\gamma}_{\alpha,\beta,w,a+} f(t) = \int_a^t e^{\gamma}_{\alpha,\beta}(w, t-\tau) f(\tau) d\tau, \tag{5}$$

where $0 \le a < t < b \le \infty$ and $f \in L^1(a,b)$, $\alpha, \beta, \lambda, w \in \mathbb{C}$, $Re(\alpha) > 0$.

Caputo-Fabrizio (CF) Operator

Caputo Fabrizio operator (see [4]) defined as:

$${}^{CF}D^{\alpha}_{a+} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t exp\Big[-\frac{\alpha}{1-\alpha}(t-\tau)\Big] f'(\tau) d\tau, \tag{6}$$

where $f \in H^1(a,b), (H^m(a,b)$ is a Hilbert space which contain function whose m^{th} derivative exists and is continuous, here we are using $H^1(a,b))$, $b > a$, $\alpha \in$ (0,1), $f'(t)$ represent the first derivative of $f(t)$, $M(\alpha)$ is a normalization constant $M(0) = M(1) = 1$.

Atangana-Baleanu (ABC) Operator

ABC operator in Caputo sense (see [3]) is defined as:

$${}^{ABC}D^{\alpha}_{a+} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\alpha}\Big[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\Big] f'(\tau) d\tau, \tag{7}$$

where B(α) has the same properties as in Caputo Fabrizio case. Where $f \in H^1(a,b)$, $b > a$, $f'(t)$ represent the first derivative of $f(t)$, $B(\alpha)$ is a normalization constant $B(0) = B(1) = 1$ and $\alpha \in$ (0,1).

Wright Hypergeometric Function

Wright hypergeometric function is defined as:

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(\lambda_i,\eta_i)_{1,p} \\ (\mu_i,\zeta_i)_{1,q}\end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda_1+\eta_1 n)\cdots\Gamma(\lambda_p+\eta_p n)}{\Gamma(\mu_1+\zeta_1 n)\cdots\Gamma(\mu_q+\zeta_q n)} \frac{z^n}{n!}, $$

where the coefficients $\eta_r(r=1,...,p)$ and $\zeta_s(s=1,...,q)$ are positive real numbers, such that $1+\sum_{s=1}^{q}\zeta_s-\sum_{r=1}^{p}\eta_r \geq 0$.

The main result in our paper is presented below. Here, we develop an extended hyperbolic function with the help of an extended beta function. Beta function have many extensions (see [2, 12]). A new *Extended Beta Function* is given by Shadab et al. [12] in 2018 using Mittag Leffler function defined as:

$$ B_p^{\iota}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{\iota}\Big(-\frac{p}{t(1-t)}\Big)dt, \tag{8} $$

where $\iota \in R_0^+$, $Re(p)>0$.

Extended Hyperbolic Function

The basic hyperbolic function is defined like

$$ h_r(z;n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1}E_{n,r}(z^n), r=1,2,... \tag{9} $$

or

$$ h_r^{\lambda}(z;n) = \sum_{k=0}^{\infty} \frac{(\lambda)_k z^{nk+r-1}}{\Gamma(nk+r)k!} = z^{r-1}E_{n,r}^{\lambda}(z^n), r=1,2,... \tag{10} $$

By using the fact that $\frac{(\lambda)_k}{(\xi)_k} = \frac{B_p^{\alpha}(\lambda+k,\xi-\lambda)}{B(\lambda,\xi-\lambda)}$, we get

$$ h_{r,\iota,p}^{\lambda,\xi}(z;n) = \sum_{k=0}^{\infty} \frac{B_p^{\iota}(\lambda+k,\xi-\lambda)(\xi)_k z^{nk+r-1}}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} = z^{r-1}E_{n,r,\alpha,p}^{\lambda,\xi}(z^n), \tag{11} $$

for $r=1,2,\ldots$ in this $\lambda,\xi \in \mathbb{C}, (Re(\lambda), Re(p)>0, Re(\xi)>0), \alpha \in R_0^+$, $Re(p) > 0$.

2. Properties of Hyperbolic Function

Integral Representation Using extended beta function we can represent integral representation of extended hyperbolic function. Here, we have discussed some properties on it:

Theorem 2.1. *Let* $\lambda,\xi \in \mathbb{C}, (Re(\lambda), Re(\xi)>0)$, $\iota \in R_0^+$, $Re(p)>0$. *Then*

$$ h_{r,\iota,p}^{\lambda,\xi}(z;n) = \frac{1}{B(\lambda,\xi-\lambda)} \int_0^1 t^{\lambda-\frac{r}{n}+\frac{(1-n)}{n}}(1-t)^{\xi-\lambda-1} E_{\iota}\Big(-\frac{p}{t(1-t)}\Big) h_r^{\xi}(zt^{\frac{1}{n}};n)dt. \tag{12} $$

Proof. Taking left hand side of equation (12) and using equation (11) and (8), we have

$$h_{r,\iota,p}^{\lambda,\xi}(z;n) = \sum_{k=0}^{\infty} \frac{(\xi)_k (z)^{nk+r-1}}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} \int_0^1 t^{\lambda+k-1}(1-t)^{\xi-\lambda-1} E_\iota\Big(-\frac{p}{t(1-t)}\Big)dt, \quad (13)$$

Multiply and divide by $t^{\frac{1}{n}(kn+r-1)}$ under the integral and changing the order of integration and summation, we get

$$h_{r,\iota,p}^{\lambda,\xi}(z;n) = \int_0^1 \frac{t^{\lambda-1-\frac{r}{n}+\frac{1}{n}}(1-t)^{\xi-\lambda-1} E_\iota\Big(-\frac{p}{t(1-t)}\Big)}{B(\lambda,\xi-\lambda)} \sum_{k=0}^{\infty} \frac{(\xi)_k (t^{\frac{1}{n}} z)^{nk+r-1}}{\Gamma(nk+r)k!} dt, \quad (14)$$

by using the definition of hyperbolic function from equation (10), we get the desired result. ■

Corollary 2.2.

$$\begin{aligned} h_{r,\iota,p}^{\lambda,\xi}(z;n) &= \frac{1}{B(\lambda,\xi-\lambda)} \int_0^\infty \frac{u^{\lambda-\frac{r}{n}+\frac{(1-n)}{n}}}{(1+u)^{\frac{-r}{n}+\frac{1}{n}+\xi}} \\ &\quad \times E_\iota\Big(-\frac{p(1+u)^2}{u}\Big) h_r^\xi\Big(z\Big(\frac{u}{1+u}\Big)^{\frac{1}{n}}; n\Big) du. \end{aligned}$$

we get the above result by substituting $t = \frac{u}{1+u}$ *in the equation* (12).

Corollary 2.3.

$$\begin{aligned} h_{r,\iota,p}^{\lambda,\xi}(z;n) &= \frac{2}{B(\lambda,\xi-\lambda)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\lambda-\frac{2r}{n}+\frac{2}{n}-1} (\cos\theta)^{2\xi-2\lambda-1} \\ &\quad \times E_\iota\Big(-\frac{p}{\sin^2\theta\cos^2\theta}\Big) h_r^\xi(z(\sin^2\theta)^{\frac{1}{n}}; n) d\theta. \end{aligned}$$

The Trigonometric form of the hyperbolic function is given by taking $t = \sin^2\theta$ *in equation* (12).

Theorem 2.4. *For hyperbolic function, the differential formula is*

$$\frac{d^q}{dz^q} h_{r,\iota,p}^{\lambda,\xi}(z;n) = h_{r-q,\iota,p}^{\lambda,\xi}(z;n),$$

$\lambda,\xi \in \mathbb{C}, Re(\lambda)$ *and* $Re(\xi) > 0, \iota \in R_0^+,\ Re(p) > 0.$

Proof. Differentiating extended hyperbolic function, we get

$$\begin{aligned} \frac{d^q}{dz^q} h_{r,\iota,p}^{\lambda,\xi}(z;n) &= \frac{d^q}{dz^q} \sum_{k=0}^{\infty} \frac{B_p^\iota(\lambda+k,\xi-\lambda)(\xi)_k z^{nk+r-1}}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!}, \\ &= \sum_{k=0}^{\infty} \frac{B_p^\iota(\lambda+k,\xi-\lambda)(\xi)_k z^{nk+r-q-1}}{B(\lambda,\xi-\lambda)\Gamma(nk+r-q)k!}. \end{aligned}$$

By applying the definition of the hyperbolic function, we get the result. ∎

Theorem 2.5. *The Mellin transformation of the extended hyperbolic function is*

$$M\Big[h_{r,\iota,p}^{\lambda,\xi}(z);s\Big]=\frac{\Gamma^{\iota}(s)\Gamma(\xi-\lambda+s)z^{r-1}}{\Gamma(\lambda)\Gamma(\xi-\lambda)}\,{}_2\Psi_2\left[\begin{matrix}(\xi,1);\ (\lambda+s,1);\\(r,n);\ (\xi+2s,1);\end{matrix}z^n\right]. \tag{15}$$

Proof. Let $F=M\Big[h_{r,\iota,p}^{\lambda,\xi}(z);s\Big]$. Applying Mellin transformation on the extended hyperbolic function, we get

$$\begin{aligned}F&=\int_0^\infty p^{s-1}h_{r,\iota,p}^{\lambda,\xi}(z;n)dp,\\&=\frac{1}{B(\lambda,\xi-\lambda)}\sum_{k=0}^{\infty}\int_0^\infty p^{s-1}\int_0^1 t^{\lambda+k-1}(1-t)^{\xi-\lambda-1}\\&\quad\times E_\iota\Big(-\frac{p}{t(1-t)}\Big)\frac{z^{nk+r-1}}{\Gamma(nk+r)k!}(\xi)_k\,dtdp,\end{aligned}$$

on changing the order of integration

$$\begin{aligned}F&=\frac{1}{B(\lambda,\xi-\lambda)}\sum_{k=0}^{\infty}\int_0^1\left\{t^{\lambda+k-1}(1-t)^{\xi-\lambda-1}\frac{z^{nk+r-1}}{\Gamma(nk+r)k!}(\xi)_k\right\}dt\\&\quad\times\int_0^\infty p^{s-1}E_\iota\Big(-\frac{p}{t(1-t)}\Big)dp,\end{aligned}$$

substituting $u=\frac{p}{t(1-t)}$, we get

$$\begin{aligned}F&=\frac{1}{B(\lambda,\xi-\lambda)}\sum_{k=0}^{\infty}\int_0^1\left\{t^{\lambda+k-1}(1-t)^{\xi-\lambda-1}\frac{z^{nk+r-1}}{\Gamma(nk+r)k!}(\xi)_k\right\}dt\\&\quad\times\int_0^\infty u^{s-1}t^s(1-t)^sE_\iota(-u)du,\\&=\frac{1}{B(\lambda,\xi-\lambda)}\sum_{k=0}^{\infty}\int_0^1\left\{t^{\lambda+k+s-1}(1-t)^{\xi-\lambda+s-1}\frac{z^{nk+r-1}}{\Gamma(nk+r)k!}(\xi)_k\right\}dt\\&\quad\times\int_0^\infty u^{s-1}E_\iota(-u)du,\end{aligned}$$

here we use the extended gamma function introduced by Pucheta [11]

$$\int_0^\infty u^{s-1}E_\iota(-u)du=\Gamma^{\iota}(s),$$

and we have

$$F = \frac{\Gamma^{\iota}(s)}{B(\lambda,\xi-\lambda)} \sum_{k=0}^{\infty} \frac{z^{nk+r-1}(\xi)_k \Gamma(\lambda+k+s)\Gamma(\xi-\lambda+s)}{\Gamma(nk+r)k!\Gamma(k+\xi+2s)},$$

$$= \frac{\Gamma^{\iota}(s) z^{r-1}}{B(\lambda,\xi-\lambda)} \sum_{k=0}^{\infty} \frac{z^{nk}(\xi)_k \Gamma(\lambda+k+s)\Gamma(\xi-\lambda+s)}{\Gamma(nk+r)k!\Gamma(k+\xi+2s)},$$

$$= \frac{\Gamma^{\iota}(s)\Gamma(\xi-\lambda+s) z^{r-1}}{\Gamma(\lambda)\Gamma(\xi-\lambda)} {}_2\Psi_2\left[\begin{matrix}(\xi,1), & (\lambda+s,1); \\ (r,n), & (\xi+2s,1);\end{matrix} z^n\right],$$

this is the desired result. ■

Corollary 2.6. *Taking $s=1$ in Theorem 2.3, we get*

$$\int_0^{\infty} h^{\lambda,\xi}_{r,\iota,p}(z;p)dp = \frac{\Gamma^{\iota}(1)\Gamma(\xi-\lambda+1) z^{r-1}}{\Gamma(\lambda)\Gamma(\xi-\lambda)} {}_2\Psi_2\left[\begin{matrix}(\xi,1), & (\lambda+1,1); \\ (r,n), & (\xi+2,1);\end{matrix} z^n\right]. \tag{16}$$

Corollary 2.7. *Taking Inverse Mellin Transformation, we get*

$$h^{\lambda,\xi}_{r,\iota,p}(z;n) = \frac{1}{2\pi i \Gamma(\lambda)\Gamma(\xi-\lambda)} \int_{\upsilon-i\infty}^{\upsilon+i\infty} \Gamma^{\iota}(s)\Gamma(\xi-\lambda+s) z^{r-1}$$
$$\times {}_2\Psi_2\left[\begin{matrix}(\xi,1); & (\lambda+s,1); \\ (r,n); & (\xi+2s,1);\end{matrix} z^n\right] p^{-s} ds,$$

where $\xi > 0$ and $Re(p) > 0$.

Theorem 2.8. *Laplace Transform of the extended hyperbolic function is*

$$L\left[h^{\lambda,\xi}_{r,\iota,p}(z;n); s\right] = \frac{1}{s^r} \sum_{k=0}^{\infty} \frac{B^{\iota}_p(\lambda+k,\xi-\lambda)(\xi)_k}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} \frac{\Gamma(kn+r)}{s^{nk}}.$$

Proof. From the definition of Laplace, we have

$$L\left[h^{\lambda,\xi}_{r,\iota,p}(z;n); s\right] = \int_0^{\infty} e^{(-sz)} h^{\lambda,\xi}_{r,\iota,p}(z;n) dz,$$

by equation (11), we get

$$L\left[h^{\lambda,\xi}_{r,\iota,p}(z;n); s\right] = \int_0^{\infty} e^{(-sz)} \sum_{k=0}^{\infty} \frac{B^{\iota}_p(\lambda+k,\xi-\lambda)(\xi)_k z^{nk+r-1}}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} dz, \tag{17}$$

$$= \sum_{k=0}^{\infty} \frac{B^{\iota}_p(\lambda+k,\xi-\lambda)(\xi)_k}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} \int_0^{\infty} e^{(-sz)} z^{nk+r-1} dz.$$

Substituting

$$\int_0^\infty e^{(-sz)} z^{nk+r-1} dz = \frac{\Gamma(kn+r)}{s^{nk+r}},$$

in the equation, we get the desired result.

$$\begin{aligned} L\left[h^{\lambda,\xi}_{r,\iota,p}(z;n);s\right] &= \sum_{k=0}^{\infty} \frac{B^{\iota}_{p}(\lambda+k,\xi-\lambda)(\xi)_k}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} \frac{\Gamma(kn+r)}{s^{nk+r}}, \\ &= \frac{1}{s^r} \sum_{k=0}^{\infty} \frac{B^{\iota}_{p}(\lambda+k,\xi-\lambda)(\xi)_k}{B(\lambda,\xi-\lambda)\Gamma(nk+r)k!} \frac{\Gamma(kn+r)}{s^{nk}}. \end{aligned}$$

■

3. Fractional Calculus of Extended Hyperbolic Function

Fractional calculus has many connections with special functions and a major application of fractional calculus within pure mathematics is to prove new relations and identities between special functions. The fractional derivatives and integrals are important aspects of fractional calculus. In this section, we applying the Prabhakar fractional integral operator, Caputo-Fabrizio operator and Atangana-Baleanu operator (ABC) on an extended hyperbolic function to get a different type of properties. In [1], it was declared that the Caputo-Fabrizio operator can be re-explained as a simple comprehension of a Prabhakar fractional integral. Indeed, if we recall that

$$E^{1,1}_{1} = exp(\theta t)$$

therefore

$$e^{1,1}_{1} = exp(\theta t)$$

then we get the following theorem.

Theorem 3.1. *Let $h^{\lambda,\xi}_{r,\iota;p}(t;n) \in L^1(a,b)$ with $b > a$ and $a, b \in R$. Then,*

$$E^{\gamma}_{\chi,\phi,\theta,a+} h^{\lambda,\xi}_{r,\iota,p}(t-a;n) = \sum_{k=0}^{\infty} (\gamma)_k \frac{\theta^k}{k!} h^{\lambda,\xi}_{\chi k+\phi+r,\iota,p}(t-a;n),$$

where $\chi, \phi \in R^+$, $\gamma, \theta, \xi, \lambda \in \mathbb{C}$, $Re(\xi) > 0$,$Re(\theta) > 0$,$Re(\gamma) > 0$, $Re(\lambda) > 0$, $\iota \in R^+_0$, $Re(p) > 0$.

Proof. Let $F = E^{\gamma}_{\chi,\phi,\theta,a+} h^{\lambda,\xi}_{r,\iota,p}(t-a;n)$. By using equation (4) and (5) in left hand side of above equation, we get

$$E^{\gamma}_{\chi,\phi,\theta,a+} h^{\lambda,\xi}_{r,\iota,p}(t-a;n) = \int_a^t (t-\tau)^{\phi-1} E^{\gamma}_{\chi,\phi}(\theta(t-\tau)^{\chi}) h^{\lambda,\xi}_{r,\iota,p}(\tau-a;n) d\tau. \quad (18)$$

Put $\tau - a = u(t-a)$. Then $d\tau = du(t-a)$. We get

$$F = \int_0^1 (t-a)^{\phi-1}(1-u)^{\phi-1} \sum_{k,s=0}^{\infty} \frac{(t-a)^{\chi k}(\gamma)_k \theta^k (1-u)^{\chi k}}{\Gamma(\chi k+\phi)k!} \times \frac{B_p^{\iota}(\lambda+s,\xi-\lambda)(\xi)_s (t-a)^{ns+r} u^{ns+r-1}}{B(\lambda,\xi-\lambda)\Gamma(ns+r)s!} du, \tag{19}$$

Change the order of summation and integration and applying the definition of the beta function, we get

$$\begin{aligned} F &= \sum_{k=0}^{\infty} \frac{(\gamma)_k \theta^k}{\Gamma(\chi k+\phi)k!} \sum_{s=0}^{\infty} \frac{B_p^{\iota}(\lambda+s,\xi-\lambda)(\xi)_s}{B(\lambda,\xi-\lambda)\Gamma(ns+r)s!} \\ &\quad \times (t-a)^{\chi k+\phi+ns+r-1} \frac{\Gamma(\chi k+\phi)\Gamma(ns+r)}{\Gamma(\chi k+\phi+ns+r)}, \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_k \theta^k}{k!} \sum_{s=0}^{\infty} \frac{B_p^{\iota}(\lambda+s,\xi-\lambda)(\xi)_s}{B(\lambda,\xi-\lambda)\Gamma(\chi k+\phi+ns+r)s!} (t-a)^{\chi k+\phi+ns+r-1}, \end{aligned}$$

by definition of the hypergeometric function, we get

$$F = \sum_{k=0}^{\infty} \frac{(\gamma)_k \theta^k}{k!} h_{\chi k+\phi+r,\iota;p}^{\lambda,\xi}(t-a;n).$$

∎

Remark 3.2. Here $AC(a,b)$ represents the class of absolutely continuous functions. For these kind of functions, it is important to recall that, if $f \in AC(a,b)$ then f is differentiable on(a,b) and $f' \in L_1(a,b)$.

Theorem 3.3. *Let $h_{r,\iota,p}^{\lambda,\xi}(t;n) \in L^1(a,b)$ with $b > a$ and $a,b \in R$. Then,*

$$E_{\chi,\phi,\theta,a+}^{\gamma} h_{r,\iota,p}^{\lambda,\xi}(t-a;n) = \sum_{k=0}^{\infty} (\gamma)_k \frac{\theta^k}{k!} \Delta_{a+}^{\chi k+\phi} h_{r,\iota,p}^{\lambda,\xi}(\tau-a;n), \tag{20}$$

provided that $\chi,\phi \in R^+$ and $\gamma,\theta \in \mathbb{C}$ and where

$$\Delta_{a+}^{\varsigma} f(t) = \frac{1}{\Gamma(\varsigma)} \int_a^t f(\tau)(t-\tau)^{\varsigma-1} d\tau, \tag{21}$$

is the Riemann-Liouville fractional integral, with $\varsigma \in R^+$.

Proof. From equation (5), we have

$$E_{\chi,\phi,\theta,a+}^{\gamma}\left[h_{r,\iota;p}^{\lambda,\xi}(t-a;n)\right] = \int_a^t e_{\chi,\phi}^{\gamma}(\theta,t-\tau) h_{r,\iota;p}^{\lambda,\xi}(t-a;n) d\tau,$$

using equation (3) and (4), in the above result

$$E_{\chi,\phi,\theta,a+}^{\gamma}\left[h_{r,\iota,p}^{\lambda,\xi}(t-a;n)\right] = \int_a^t \sum_{k=0}^{\infty} (t-\tau)^{\phi-1} \frac{(\gamma)_k \theta^k (t-\tau)^{\chi k}}{\Gamma(\chi k+\phi)k!} h_{r,\iota;p}^{\lambda,\xi}(\tau-a;n) d\tau.$$

Now changing the order of summation and integration and applying the definition of Riemann-Liouville fractional integral, we get the required result. ■

Theorem 3.4. *Let* $h^{\lambda,\xi}_{r,\iota,p}(t;n) \in AC(a,b)$ *with* $b > a$, $a, b \in R$ *and let* $0 < \alpha < 1$. *Then*

$$\begin{aligned} {}^{CF}D^{\alpha}_{a+}(h^{\lambda,\xi}_{r,\iota;p}(t-a;n)) &= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k h^{\lambda,\xi}_{k+r,\iota,p}(t-a;n) \\ &= \frac{M(\alpha)}{1-\alpha}E^{1}_{1,0,\theta,a+}h^{\lambda,\xi}_{r,\iota,p}(t-a;n), \\ &= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k\left[\Delta^k_{a+}h^{\lambda,\xi}_{r,\iota,p}(z-a,n)\right], \end{aligned} \tag{22}$$

here, $M(\alpha)$ *is a normalization constant such that* $M(0) = M(1) = 1$ *and* $0 < \alpha < 1$, *where* $\theta(\alpha) = -\frac{\alpha}{1-\alpha}$.

Proof. Let $F = {}^{CF}D^{\alpha}_{a+}(h^{\lambda,\xi}_{r,\iota;p}(t-a;n))$. With the help of equation (6) and the certainty that $e^1_{1,1}(\theta;t) = E^1_{1,1}(\theta t) = exp(\theta t)$, we have

$$\begin{aligned} F &= \frac{M(\alpha)}{1-\alpha}\int_a^t exp\Big[-\frac{\alpha}{1-\alpha}(t-\tau)\Big]h'^{\lambda,\xi}_{r,\iota,p}(\tau-a;n)d\tau, \\ &= \frac{M(\alpha)}{1-\alpha}\int_a^t E^1_{1,1}\Big[-\frac{\alpha}{1-\alpha}(t-\tau)\Big]h'^{\lambda,\xi}_{r,\iota;p}(\tau-a;n)d\tau, \\ &= \frac{M(\alpha)}{1-\alpha}\int_a^t e^1_{1,1}\Big[-\frac{\alpha}{1-\alpha};(t-\tau)\Big]h'^{\lambda,\xi}_{r,\iota,p}(\tau-a;n)d\tau. \end{aligned}$$

After differentiating extended hyperbolic function w.r.t τ and substituting $\tau - a = u(t-a)$, using definition of the beta function, we have

$$\begin{aligned} F &= \frac{M(\alpha)}{1-\alpha}\sum_{k,s=0}^{\infty}\frac{(1)_k\theta^k}{\Gamma(k+1)k!}\frac{B^{\iota}_p(\lambda+s,\xi-\lambda)(\xi)_s(ns+r-1)}{B(\lambda,\xi-\lambda)\Gamma(ns+r)s!} \\ &\quad\times\frac{\Gamma(k+1)\Gamma(ns+r-1)(t-a)^{ns+k+r-1}}{\Gamma(ns+k+r)}, \\ &= \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k h^{\lambda,\xi}_{k+r,\iota,p}(t-a;n). \end{aligned}$$

Again using equation (20) and (23), we get the result in terms of Riemann-Liouville fractional integral operator. ■

Theorem 3.5. *Let* $h^{\lambda,\xi}_{r,\iota,p}(t;n) \in AC(a,b)$ *with* $b > a$, $a, b \in R$ *and let* $0 < \alpha < 1$.

Then

$$\begin{aligned} {}^{ABC}D^{\alpha}_{a+}(h^{\lambda,\xi}_{r,\iota,p}(t-a;n)) &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k h^{\lambda,\xi}_{\alpha k+r,\iota,p}(t-a;n) \\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k\left[\Delta^{\alpha k}_{a^+}h^{\lambda,\xi}_{r,\iota,p}(z-a,n)\right], \qquad (23) \end{aligned}$$

where $\theta(\alpha) = -\frac{\alpha}{1-\alpha}$.

Proof. Let $F = {}^{ABC}\, D^{\alpha}_{a+}(h^{\lambda,\xi}_{r,\iota,p}(t-a;n))$. If we consider the definition by equation (7) and apply the certainty that $e^{1}_{\alpha,1}(\theta;t) = E^{1}_{\alpha,1}(\theta t)$,

$$F = \frac{B(\alpha)}{1-\alpha}\int_a^t E_\alpha\Big[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\Big]h'^{\lambda,\xi}_{r,\iota,p}(\tau-a;n)d\tau,$$

by equation (1) and (11), we get

$$\begin{aligned} F &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\theta^k}{\Gamma(\alpha k+1)}\int_a^t (t-\tau)^{\alpha k+1-1}\sum_{s=0}^{\infty}\frac{E(ns+r-1)}{B(\lambda,\xi-\lambda)\Gamma(ns+r)s!}d\tau, \\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\theta^k}{\Gamma(\alpha k+1)}\int_a^t (t-\tau)^{\alpha k+1-1}\sum_{s=0}^{\infty}\frac{E}{B(\lambda,\xi-\lambda)\Gamma(ns+r-1)}d\tau, \end{aligned}$$

for $E = B^{\iota}_{p}(\lambda+s,\xi-\lambda)(\xi)_s(\tau-a)^{ns+r-2}$. From equation (24)

$$F = \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\theta^k\left[\Delta^{\alpha k+1}_{a^+}h^{\lambda,\xi}_{r-1,\iota,p}(t-a,n)\right],$$

follow the same procedure as the above theorem, we get the result in terms of Riemann-Liouville fractional integral. ∎

4. Graphical Representation of Hyperbolic Function for Different Values of α

In this section, we have showing the graphical representation of hyperbolic function with certain assumptions with different value of α=0.6, 0.7, 0.8, 0.9.

5. Interpretation of Graphs

Figure-1 represents the variations of extended hyperbolic function with respect to z at different assumed values of α between 0.6 to 0.9.

(i) At $\alpha = 0.6$ in positive z-scale, function decreases between 0 to 80 but beyond 80 it will start increasing and it is fluctuating in negative z-scale.

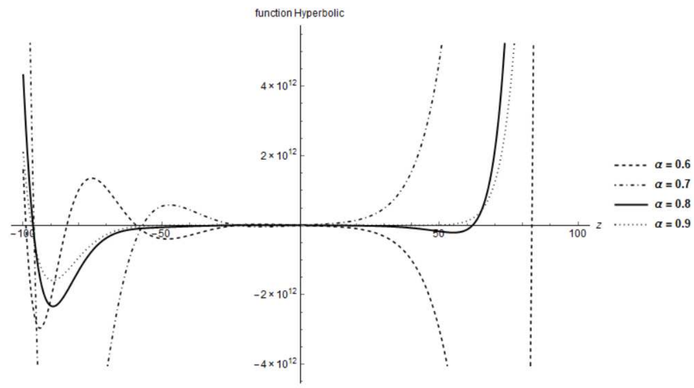

Figure 1: λ=0.00501, ω=3 ,p=3, r=2, n=3

Values of Table

z	$h^{\lambda,\xi}_{r,0.6,p}(z,n)$	$h^{\lambda,\xi}_{r,0.7,p}(z,n)$	$h^{\lambda,\xi}_{r,0.8,p}(z,n)$	$h^{\lambda,\xi}_{r,0.9,p}(z,n)$
-100	1.59542*10^{12}	1.83845*10^{13}	4.32533*10^{12}	2.10434*10^{12}
-80	1.01462*10^{12}	-1.06145*10^{13}	-1.30369*10^{12}	-9.30888*10^{11}
-60	5.54346*10^{10}	-5.4566*10^{11}	-1.08468*10^{11}	-4.57691*10^{10}
-40	-2.88201*10^{11}	4.44973*10^{11}	-2.89441*10^{10}	2.06831*10^{8}
-20	1.57555*10^{10}	-2.3183*10^{10}	2.33464*10^{9}	8.61138*10^{7}
0	0.	0.	0.	0.
20	-9.44505*10^{10}	1.44406*10^{11}	-9.09445*10^{9}	-1.0054*10^{8}
40	-1.00528*10^{12}	1.55142*10^{12}	-7.66077*10^{10}	, -2.10245*10^{8}
60	-8.08769*/10^{12}	1.38676*10^{13}	-1.21964*10^{11}	1.31356*10^{11}
80	-1.72821*10^{13}	1.21068*10^{14}	2.08211*10^{13}	9.45135*10^{12}
100	1.24932*10^{15}	, 1.47142*10^{15}	, 7.81522*10^{14}	, 3.44487*10^{14}

(ii) when $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ +ve in positive z-scale, function continuously increasing upto infinity and for negative z-scale it is fluctuating.

Figure-2,3 and Figure-4 represents the comparison between Caputo Fabrizio operator and Atangana Baleanu operator with respect to t at different value of r and fixed value of $\alpha = 0.8$.

(i) In all the three curves Atangana Baleanu operator exists only for positive t values, where as Caputo Fabrizio operator exists for entire t scale.

(ii) At r=10 and 20, the Caputo Fabrizio operator shows the symmetric behaviour about the origin and Atangana Baleanu operator shows similar behaviour in positive octant.

(iii) At r=15, Caputo Fabrizio operator shows approximately similar behaviour for entire t scale and attained positive values only, with a flate board

Graphical comparison of CAPUTO-FABRIZIO (CF) operator and ATANGANA-BALEANU (ABC) operator of extended hyperbolic function defined by equation (7) for different values of r

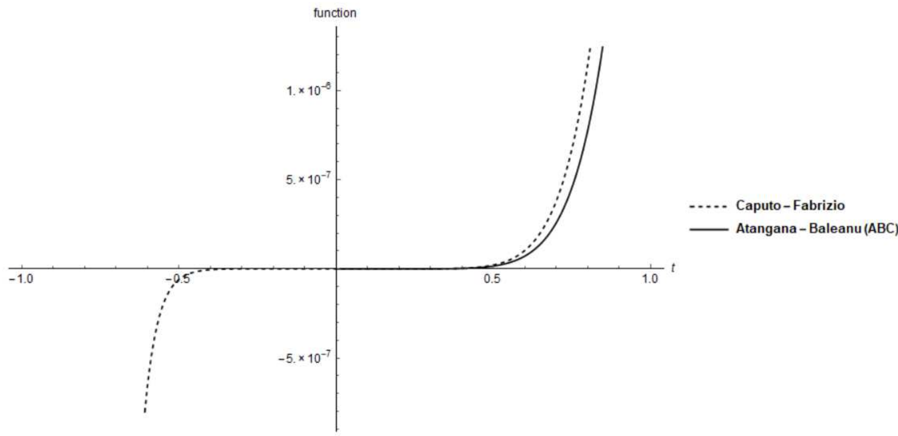

Figure 2: α=0.8, $M(\alpha) = 1$, $B(\alpha) = 1, r = 10, n = 1, a = 0$

Values of Table-1

t	${}^{CF}D_0{}^{\lambda,\omega}h_r(z,n)$	${}^{ABC}D_0{}^{\alpha}h_r(z,n)$
-1	-0.00253328	-0.00536474 + 0.0154487 I
-0.8	-0.0000567934	-0.00031925 + 0.000169886 I
-0.6	-6.5736*10^{-7}	-3.30594*10^{-6} - 7.17875*10^{-7} I
-0.4	-5.10262*10^{-9}	-7.82417*10^{-9} - 3.49087*10^{-9} I
-0.2	-7.69054*10^{-12}	-8.25695*10^{-12} + 8.63312*10^{-13} I
0	6.89651*10^{-152}	6.89651*10^{-152}
0.2	6.52474*10^{-12}	5.5977*10^{-12}
0.4	3.05809*10^{-9}	2.31852*10^{-9}
0.6	1.03544*10^{-7}	7.19255*10^{-8}
0.8	1.14868*10^{-6}	7.78484*10^{-7}
1	6.72151*10^{-6}	4.77874*10^{-6}

base and Atangana Baleanu operator follows Caputo Fabrizio operator in positive quadrant as it has imaginary values for negative values of t. So we can say that the Atangana Baleanu operator behaves like a modulus of Caputo Fabrizio operator.

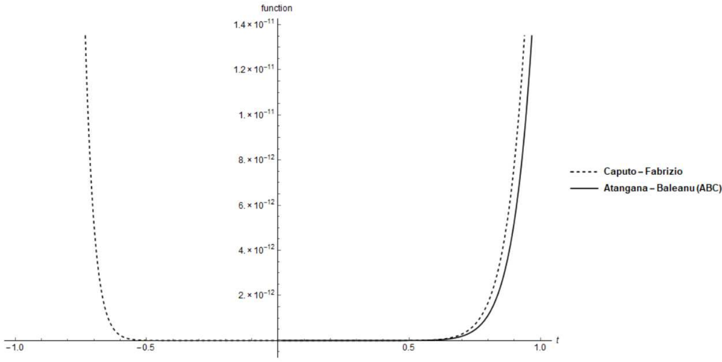

Figure 3: α=0.8, $M(\alpha) = 1$, $B(\alpha) = 1, r = 15, n = 1, a = 0$

Values of Table-2

t	${}^{CF}D_0{}^{\lambda,\omega}h_r(z,n)$	${}^{ABC}D_0{}^{\alpha}h_r(z,n)$
-1	1.18892*10^{-}8	2.54706*10^{-8} - 7.29653*10^{-8} I
-0.8	8.66783*10^{-11}	4.94987*10^{-10} - 2.61943*10^{-10} I
-0.6	2.27772*10^{-13}	1.20671*10^{-12} + 2.77072*10^{-13} I
-0.4	2.1144*10^{-16}	3.42364*10^{-16} + 1.9231*10^{-16} I
-0.2	9.96342*10^{-21}	1.06402*10^{-20} - 7.19192*10^{-22} I
0	1.51316*10^{-238}	1.51316*10^{-238}
0.2	8.91133*10^{-}21	7.70316*10^{-21}
0.4	1.36467*10^{-16}	1.03481*10^{-16}
0.6	3.56411*10^{-14}	2.4538*10^{-14}
0.8	1.68474*10^{-12}	1.1225*10^{-12}
1	3.03202*10^{-11}	2.10463*10^{-11}

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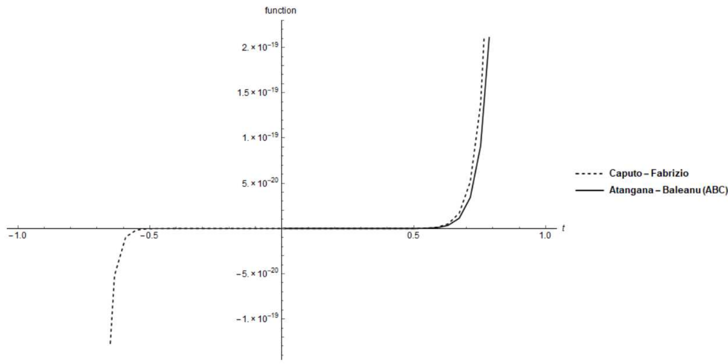

Figure 4: α=0.8, $M(\alpha) = 1,\ B(\alpha) = 1, r = 20, n = 1, a = 0$

Values of Table-3

t	${}^{CF}D_0{}^{\lambda,\omega}h_r(z,n)$	${}^{ABC}D_0{}^{\alpha}h_r(z,n)$
-1	-9.08492*10^{-15}	-1.95927*10^{-14} + 5.58329*10^{-14} I
-0.8	-2.16382*10^{-17}	-1.24327*10^{-16} + 6.55039*10^{-17} I
-0.6	-1.32256*10^{-20}	-7.16186*10^{-20} - 1.687*10^{-20} I
-0.4	-1.53524*10^{-24}	-2.55248*10^{-24} - 1.58165*10^{-24} I
-0.2	-2.25401*10^{-30}	-2.39835*410^{-30} + 1.17548*10^{-31} I
0	5.716100183620844*10^{-326}	5.716100183620539*10^{-326}
0.2	2.06975*10^{-30}	1.79648*10^{-30}
0.4	1.02528*10^{-24}	7.7773*10^{-25}
0.6	2.05009*10^{-21}	1.4054*10^{-21}
0.8	4.10758*10^{-19}	2.71383*10^{-19}
1	2.26532*10^{-17}	1.55385*10^{-17}

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