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Reproducing Kernel for Periodic Boundary Conditions

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Abstract. In this paper, we introduced a reproducing kernel space which is a particular class of Hilbert space. We discuss various properties of the reproducing kernel. In particular, our aim is to construct kernel in reproducing kernel Hilbert space of the specific function space (Sobolev space) with the improved inner product and norm. Also, we derive the reproducing kernel for periodic boundary conditions.

Keywords: Inner product; Hilbert space; Reproducing kernel Hilbert space; Reproducing kernel; Periodic boundary conditions.

1. Introduction

Reproducing kernels were discovered during the initial stage of the twentieth century by Zeremba [20] in that effort the center of interest on harmonic function with boundary value. This was the earliest reproducing kernel with the reproducibility proved correlated with function family. Actually, in the early establishment develop of the reproducing kernel hypothesis, almost all the works were executed by Bergman [10, 11, 12, 13, 14], and most of the kernels discussed in the 1930's and 1940's are Bergman kernels. Bergman raise the conversation of the kernels with one or several variables to the harmonic functions, and utilized to solve Laplace equation. It can be stated that this is the establishment of a particular trend of reproducing kernel. Next development of the reproducing kernel theory was pushed by Mercer [19]. He invented the positive definite property of reproducing kernel and known its as positive definite Hermition matrix:

$$\sum_{i,j=1}^{n} k(x_i, x_j)\zeta_i\zeta_j \geq 0. \tag{1}$$

In 1950, N. Aronszajn [1] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. In this theory unifying the Bergman and Marces concept of reproducing kernel development.

Subsequently, reproducing kernel theory was use by mathematician, scientist [17, 16, 2, 7, 5, 6, 8, 9, 3, 4, 18] like to solve the theoretical problems of many special fields. In 1986, Cui [15] construct the reproducing kernel space and corresponding kernel in the Sobolev space.

Here, we review some aspect of reproducing kernel space and then construct the reproducing kernel for the inner product and norm of Sobolev space for $m = 2$ with periodic boundary conditions.

2. Preliminaries

Definition 2.1. *An inner product space is a vector space V over the field $\mathbb{F}$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$) together with an inner product*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}, \tag{2}$$

that satisfies the following three properties for all vectors $\varrho, \varphi, \vartheta \in V$ and all scalars $a \in F$:

(i) $\langle \varrho, \varphi \rangle = \overline{\langle \varphi, \varrho \rangle}$.
(ii) $\langle a\varrho, \varphi \rangle = a\langle \varrho, \varphi \rangle$.
(iii) $\langle \varrho + \varphi, \vartheta \rangle = \langle \varrho, \vartheta \rangle + \langle \varphi, \vartheta \rangle$.
(iv) $\langle \varrho, \varphi \rangle \geq 0$.

Definition 2.2. *A Hilbert space $\mathcal{H}$ is a complete real or complex inner product space.*

Definition 2.3. *Consider $\mathcal{H} = \{f(\varrho) : f(\varrho) \in \mathbb{R}$ or $f(\varrho) \in \mathbb{C}$, ϱ is in abstract set$\}$ is endowed with $\langle f(\varrho), g(\varrho) \rangle_{\mathcal{H}}$, with respect to which $\mathcal{H}$ is a Hilbert space.*

For an abstract set X, a function $\Re(\varrho, \varphi) : X \times X \to \mathbb{F}$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$) is called the reproducing kernel of Hilbert space $\mathcal{H}$ if its satisfies,

$$\langle f(\varrho), \Re(\varrho, \varphi) \rangle_{\mathcal{H}} = f(\varphi), \tag{3}$$

for each fixed $\varphi \in X$.

Definition 2.4. *A periodic boundary conditions have equal function values or its derivatives values at end points.*

Lemma 2.5. *In reproducing kernel space $\mathcal{H}$, $\Re(\varrho,\varphi) = \overline{\Re(\varphi,\varrho)}$.*

Proof. We have

$$\Re(\varrho,\varphi) = \langle \Re(\cdot,\varphi), \Re(\cdot,\varrho)\rangle_{\mathcal{H}} = \overline{\langle \Re(\cdot,\varrho), \Re(\cdot,\varphi)\rangle_{\mathcal{H}}} = \overline{\Re(\varphi,\varrho)}.$$

Hence, $\Re(\varrho,\varphi)$ is conjugate symmetric. ∎

Lemma 2.6. *The reproducing kernel $\Re(\varrho,\varphi)$ is unique in reproducing kernel space $\mathcal{H}$.*

Proof. Let $\Phi(\varrho,\varphi)$ be also reproducing kernel. Then

$$\Phi(\varrho,\varphi) = \langle \Phi(\cdot,\varphi), \Re(\cdot,\varrho)\rangle_{\mathcal{H}} = \overline{\langle \Re(\cdot,\varrho), \Phi(\cdot,\varphi)\rangle_{\mathcal{H}}} = \overline{\Re(\varphi,\varrho)} = \Re(\varrho,\varphi).$$

Hence, reproducing kernel is unique. ∎

Lemma 2.7. *If $\Re(\varrho,\varphi)$ is the reproducing kernel in $\mathcal{H}$, then for each $\varrho \in X$, $\Re(\varrho,\varrho) \geq 0$ and $\Re(\varrho,\varrho) = 0$ if and only if $\mathcal{H} = \{0\}$.*

Proof. We have

$$\Re(\varrho,\varrho) = \langle \Re(\cdot,\varrho), \Re(\cdot,\varrho)\rangle_{\mathcal{H}} = \|\Re(\cdot,\varrho)\|^2_{\mathcal{H}}.$$

Which gives $\Re(\varrho,\varrho) \geq 0$ and $\Re(\varrho,\varrho) = 0$ if and only if $\mathcal{H} = \{0\}$. ∎

Lemma 2.8. *Reproducing kernel $\Re(\varrho,\varphi)$ is a positive definite.*

Proof. For any complex number ζ_i

$$\begin{aligned}
\sum_{\imath,\jmath=1}^{n} \overline{\zeta_i}\zeta_j \Re(\varrho_i,\varrho_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\zeta_i}\zeta_j \langle \Re(\cdot,\varrho_i), \Re(\cdot,\varrho_j)\rangle_{\mathcal{H}} \\
&= \langle \sum_{j=1}^{n} \zeta_j \Re(\cdot,\varrho_j), \sum_{i=1}^{n} \zeta_i \Re(\cdot,\varrho_i)\rangle_{\mathcal{H}} \\
&= \langle \sum_{i=1}^{n} \zeta_i \Re(\cdot,\varrho_i), \sum_{i=1}^{n} \zeta_i \Re(\cdot,\varrho_i)\rangle_{\mathcal{H}} \\
&= \| \sum_{i=1}^{n} \zeta_i \Re(\cdot,\varrho_i)\|^2_{\mathcal{H}} \geq 0.
\end{aligned}$$

Hence, reproducing kernel is positive definite. ∎

Lemma 2.9. *For any fixed $\varrho \in X$, the linear functional $\Im(f(\varrho)) = f(\varrho)$ is bounded if and only if Hilbert space $\mathcal{H}$ is a reproducing kernel space.*

Proof. Since $\mathcal{H}$ is a reproducing kernel space, there exists a reproducing kernel $\Re(\varrho, \varphi)$.

$$\begin{aligned}|\Im(f(\varrho))| = |f(\varrho)| &= |\langle f(\cdot), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}| \\ &\leq \|f(\cdot)\|_{\mathcal{H}} \|\Re(\cdot, \varrho)\|_{\mathcal{H}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\langle \Re(\cdot, \varrho), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\Re(\varrho, \varrho)}.\end{aligned}$$

Therefore, $\Im(f(\varrho)) = f(\varrho)$ is bounded.

Now, for every $f(\varrho) \in \mathcal{H}$, because of linear functional, by F. Riesz theorem there exists a unique $\Re(\cdot, \varrho) \in \mathcal{H}$, whence $f(\varrho) = \Im(f(\varrho)) = \langle f(\cdot), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}$. Hence, the lemma is proved. ■

3. Reproducing Kernel Space $\mathcal{W}_2^m[\alpha, \beta]$

In this section, the function space $\mathcal{W}_2^m[\alpha, \beta] = \{f(\varrho) : f^{(m-1)}(\varrho)$ is absolutely continuous,$f^{(m)}(\varrho) \in L^2[\alpha, \beta], \varrho \in [\alpha, \beta]\}$. For any functions $f(\varrho), g(\varrho) \in \mathcal{W}_2^m[\alpha, \beta]$, we have

$$\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1}\left[\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i g(\alpha)}{d\varrho^i} + \frac{d^i f(\beta)}{d\varrho^i}\frac{d^i g(\beta)}{d\varrho^i}\right] + \int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m g(\varrho)}{d\varrho^m} d\varrho, \quad (4)$$

$$\|f(\varrho)\|_{\mathcal{W}_2^m} = \sqrt{\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}}. \quad (5)$$

Theorem 3.1. *The space $\mathcal{W}_2^m[\alpha, \beta]$ is an inner product space.*

Proof. Let $f(\varrho), g(\varrho), h(\varrho) \in \mathcal{W}_2^m[\alpha, \beta]$. Here,

$$\begin{aligned}\langle f(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m} = &\sum_{i=0}^{m-1}\left[\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right] \\ &+ \int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho.\end{aligned}$$

Since, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 > 0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2 > 0, 0 \leq i \leq m-1$. Also, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 = 0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2 = 0$ if and only if $\frac{d^i f(\alpha)}{d\varrho^i} = 0$ and $\frac{d^i f(\beta)}{d\varrho^i} = 0$, $0 \leq i \leq m-1$. And, $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 > 0$ and $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 = 0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m} = 0, \forall \varrho \in [\alpha, \beta]$. Therefore, $\int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho > 0$ and $\int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho = 0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m} = 0, \forall \varrho \in [\alpha, \beta]$. Thus, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is positive definite. Clearly, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m} = \langle g(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m}$ which gives $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is symmetric.

Now for linearity consider scalars a and b,

$$\begin{aligned}\langle af(\varrho)+bg(\varrho),h(\varrho)\rangle_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1}\left[\left(a\frac{d^if(\alpha)}{d\varrho^i}+b\frac{d^ig(\alpha)}{d\varrho^i}\right)\frac{d^ih(\alpha)}{d\varrho^i}\right.\\ &\quad\left.+\left(a\frac{d^if(\beta)}{d\varrho^i}+b\frac{d^ig(\beta)}{d\varrho^i}\right)\frac{d^ih(\beta)}{d\varrho^i}\right]\\ &\quad+\int_\alpha^\beta\left[a\frac{d^mf(\varrho)}{d\varrho^m}+b\frac{d^mg(\varrho)}{d\varrho^m}\right]\frac{d^mh(\varrho)}{d\varrho^m}d\varrho\\ &=\sum_{i=0}^{m-1}\left[a\frac{d^if(\alpha)}{d\varrho^i}\frac{d^ih(\alpha)}{d\varrho^i}+a\frac{d^if(\beta)}{d\varrho^i}\frac{d^ih(\beta)}{d\varrho^i}\right]\\ &\quad+\int_\alpha^\beta a\frac{d^mf(\varrho)}{d\varrho^m}\frac{d^mh(\varrho)}{d\varrho^m}d\varrho\\ &\quad+\sum_{i=0}^{m-1}\left[b\frac{d^if(\alpha)}{d\varrho^i}\frac{d^ih(\alpha)}{d\varrho^i}+b\frac{d^if(\beta)}{d\varrho^i}\frac{d^ih(\beta)}{d\varrho^i}\right]\\ &\quad+\int_\alpha^\beta b\frac{d^mf(\varrho)}{d\varrho^m}\frac{d^mh(\varrho)}{d\varrho^m}d\varrho\\ &=a\langle f(\varrho),h(\varrho)\rangle_{\mathcal{W}_2^m}+b\langle g(\varrho),h(\varrho)\rangle_{\mathcal{W}_2^m}.\end{aligned}$$

Thus, $\langle f(\varrho),g(\varrho)\rangle_{\mathcal{W}_2^m}$ is linear. This completes the proof. ■

Theorem 3.2. *The space $\mathcal{W}_2^m[\alpha,\beta]$ is a Hilbert space.*

Proof. Consider $f_n(\varrho)$, $n=1,2,\ldots$ is a Cauchy sequence in $\mathcal{W}_2^m[\alpha,\beta]$. Therefore,

$$\begin{aligned}\|f_{n+p}-f_n\|^2_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1}\left[\left(\frac{d^if_{n+p}(\alpha)}{d\varrho^i}-\frac{d^if_n(\alpha)}{d\varrho^i}\right)^2\right.\\ &\quad\left.+\left(\frac{d^if_{n+p}(\beta)}{d\varrho^i}-\frac{d^if_n(\beta)}{d\varrho^i}\right)^2\right]\\ &\quad+\int_\alpha^\beta\left(\frac{d^mf_{n+p}(\varrho)}{d\varrho^m}-\frac{d^mf_n(\varrho)}{d\varrho^m}\right)^2d\varrho\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

Which gives, $\frac{d^if_{n+p}(\alpha)}{d\varrho^i}-\frac{d^if_n(\alpha)}{d\varrho^i}\to 0$ as $n\to\infty$, $0\le i\le m-1$,$n=1,2,\ldots$.

Similarly, $\frac{d^if_{n+p}(\beta)}{d\varrho^i}-\frac{d^if_n(\beta)}{d\varrho^i}\to 0$ as $n\to\infty$, $0\le i\le m-1$,$n=1,2,\ldots$. And

$$\int_\alpha^\beta\left(\frac{d^mf_{n+p}(\varrho)}{d\varrho^m}-\frac{d^mf_n(\varrho)}{d\varrho^m}\right)^2d\varrho\to 0\quad\text{as}\quad n\to\infty.$$

This indicates that for any i $(0\le i\le m-1)$, the sequence $\frac{d^if_n(\alpha)}{d\varrho^i}$ and $\frac{d^if_n(\beta)}{d\varrho^i}$,$n=1,2,\ldots$ are Cauchy sequences in R and $\frac{d^mf_n(\varrho)}{d\varrho^m}$,$n=1,2,\ldots$ is a Cauchy sequence in space $L^2[\alpha,\beta]$.

So, there exists unique real numbers c_i and d_i, $0 \le i \le m-1$ and unique function $h(\varrho) \in L^2[\alpha,\beta]$ such that, $\frac{d^i f_n(\alpha)}{d\varrho^i} \to c_i$ and $\frac{d^i f_n(\beta)}{d\varrho^i} \to d_i$, $0 \le i \le m-1$ and $\int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho)\right)^2 d\varrho \to 0$ as $n \to \infty$. We must have $g(\varrho) \in \mathcal{W}_2^m[\alpha,\beta]$ with $\frac{d^i g(\alpha)}{d\varrho^i} = c_i$, $\frac{d^i g(\beta)}{d\varrho^i} = d_i$, $0 \le i \le m-1$ and $\frac{d^m g(\varrho)}{d\varrho^m} = h(\varrho)$.

Moreover,

$$\begin{aligned}
\|f_n(\varrho) - g(\varrho)\|^2_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1} \left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - \frac{d^i g(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - \frac{d^i g(\beta)}{d\varrho^i}\right)^2\right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - \frac{d^m g(\varrho)}{d\varrho^m}\right)^2 d\varrho \\
&= \sum_{i=0}^{m-1} \left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - c_i\right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - d_i\right)^2\right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho)\right)^2 d\varrho \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, the function space $\mathcal{W}_2^m$ is a Hilbert space. ■

Theorem 3.3. *The space $\mathcal{W}_2^m[\alpha,\beta]$ is a reproducing kernel Hilbert space.*

Proof. As per Lemma 2.9, suppose that $\Im(f) = f(\varrho)$, $\varrho \in [\alpha,\beta]$ is linear functional of $\mathcal{W}_2^m[\alpha,\beta]$ and $f(\varrho) \in \mathcal{W}_2^m$. We have,

$$\begin{aligned}
\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} &= \frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} + \int_\alpha^\varrho \frac{d^m f(\varrho)}{d\varrho^m} d\varrho, \\
\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} &= \int_\varrho^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}.
\end{aligned}$$

Therefore,

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \frac{1}{2}\left[\frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}\right] + \frac{1}{2}\int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho.$$

Obviously,

$$\left|\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}}\right| \le \left|\frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}}\right| + \left|\frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}\right| + \int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right| d\varrho. \tag{6}$$

Since,

$$\begin{aligned}
&\int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right| d\varrho \\
&\le \left[(\beta - \alpha)\int_\alpha^\beta \left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2 d\varrho\right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= K_0 \left[\int_\alpha^\beta \left| \frac{d^m f(\varrho)}{d\varrho^m} \right|^2 d\varrho \right]^{\frac{1}{2}} \\
&\le K_0 \left[\sum_{i=0}^{m-1} \left(\left(\frac{d^i f(\alpha)}{d\varrho^i} \right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i} \right)^2 \right) + \int_\alpha^\beta \left| \frac{d^m f(\varrho)}{d\varrho^m} \right|^2 \right]^{\frac{1}{2}} \\
&= K_0 \|f\|_{\mathcal{W}_2^m}.
\end{aligned}
$$

Now, for any i, $0 \le i \le m-1$,

$$
\begin{aligned}
\left| \frac{d^i f(\alpha)}{d\varrho^i} \right| &\le \left[\sum_{i=0}^{m-1} \left(\left(\frac{d^i f(\alpha)}{d\varrho^i} \right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i} \right)^2 \right) + \int_\alpha^\beta \left| \frac{d^m f(\varrho)}{d\varrho^m} \right|^2 \right]^{\frac{1}{2}} \\
&= \|f\|_{\mathcal{W}_2^m}.
\end{aligned}
\tag{7}
$$

Similarly,

$$
\left| \frac{d^i f(\beta)}{d\varrho^i} \right| \le \|f\|_{\mathcal{W}_2^m}. \tag{8}
$$

From (6) to (8),

$$
\left| \frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} \right| \le K_1 \|f\|_{\mathcal{W}_2^m}. \tag{9}
$$

Analogously,

$$
\left| \frac{d^{m-2} f(\varrho)}{d\varrho^{m-2}} \right| \le K_2 \|f\|_{\mathcal{W}_2^m}. \tag{10}
$$

Thus,$|\mathfrak{I}(f)| = |f(\varrho)| \le K_m \|f\|_{\mathcal{W}_2^m}$. Hence, $\mathfrak{I}$ is bounded functional which provide that $\mathcal{W}_2^m[a,b]$ is reproducing kernel Hilbert space. ■

4. Method to Construct Reproducing Kernel

Suppose $\Re(\varrho,\varphi)$ is the reproducing kernel function of $\mathcal{W}_2^m[\alpha,\beta]$. Then for any fixed $\varphi \in [\alpha,\beta]$ and any $f(\varrho) \in \mathcal{W}_2^m[\alpha,\beta]$, $\Re(\varrho,\varphi)$ must satisfy

$$
\langle f(\varrho), \Re(\varrho,\varphi) \rangle_{\mathcal{W}_2^m} = f(\varphi). \tag{11}
$$

Therefore,

$$
\begin{aligned}
\langle f(\varrho), \Re(\varrho,\varphi) \rangle_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1} \left[\frac{d^i f(\alpha)}{d\varrho^i} \frac{\partial^i \Re(\alpha,\varphi)}{\partial\varrho^i} + \frac{d^i f(\beta)}{d\varrho^i} \frac{\partial^i \Re(\beta,\varphi)}{\partial\varrho^i} \right] \\
&\quad + \int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m} \frac{\partial^m \Re(\varrho,\varphi)}{\partial\varrho^m} d\varrho.
\end{aligned}
\tag{12}
$$

Since,

$$\int_{\alpha}^{\beta} \frac{d^m f(\varrho)}{d\varrho^m} \frac{\partial^m \Re(\varrho,\varphi)}{\partial \varrho^m} d\varrho = \sum_{i=0}^{m-1} \left((-1)^i \frac{d^{m-i-1} f(\varrho)}{d\varrho^{m-i-1}} \frac{\partial^{m+i} \Re(\varrho,\varphi)}{\partial \varrho^{m+i}} \right)_{\varrho=\alpha}^{\beta} + (-1)^m \int_{\alpha}^{\beta} f(\varrho) \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} d\varrho. \tag{13}$$

Also,

$$\sum_{i=0}^{m-1} ((-1)^i \frac{d^{m-i-1} f(\varrho)}{d\varrho^{m-i-1}} \frac{\partial^{m+i} \Re(\varrho,\varphi)}{\partial \varrho^{m+i}} = \sum_{i=0}^{m-1} (-1)^{m-i-1} \frac{d^i f(\varrho)}{d\varrho^i} \frac{\partial^{2m-i-1} \Re(\varrho,\varphi)}{\partial \varrho^{2m-i-1}}. \tag{14}$$

From equations (12) to (14), we get

$$\begin{aligned}\langle f(\varrho), \Re(\varrho,\varphi) \rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1} \Bigg[& \frac{d^i f(\alpha)}{d\varrho^i} \left(\frac{\partial^i \Re(\alpha,\varphi)}{\partial \varrho^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\alpha,\varphi)}{\partial \varrho^{2m-i-1}} \right) \\ & + \frac{d^i f(\beta)}{d\varrho^i} \left((-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\beta,\varphi)}{\partial \varrho^{2m-i-1}} + \frac{\partial^i \Re(\beta,\varphi)}{\partial \varrho^i} \right) \Bigg] \\ & + (-1)^m \int_{\alpha}^{\beta} f(\varrho) \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} d\varrho. \end{aligned} \tag{15}$$

Now, from equations (11), (15) and the Dirac delta function

$$(-1)^m \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} = \delta(\varrho - \varphi), \tag{16}$$

$$\frac{\partial^i \Re(\alpha,\varphi)}{\partial \varrho^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\alpha,\varphi)}{\partial \varrho^{2m-i-1}} = 0, \quad 0 \le i \le m-1, \tag{17}$$

$$(-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\beta,\varphi)}{\partial \varrho^{2m-i-1}} + \frac{\partial^i \Re(\beta,\varphi)}{\partial \varrho^i} = 0, \quad 0 \le i \le m-1. \tag{18}$$

Here, $\Re(\varrho,\varphi)$ is the solution of the following constant coefficient $2m$ order differential equation with boundary conditions (17) and (18)

$$(-1)^m \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} = 0. \tag{19}$$

The equation (19) has characteristic equation $\lambda^{2m} = 0$ whose the characteristic root $\lambda = 0$ with multiplicity $2m$. Therefore,

$$\Re(\varrho,\varphi) = \begin{cases} \Re_1(\varrho,\varphi) = \sum_{i=1}^{2m} c_i(\varphi) \varrho^{i-1}, & \varrho < \varphi \\ \Re_2(\varrho,\varphi) = \sum_{i=1}^{2m} d_i(\varphi) \varrho^{i-1}, & \varrho > \varphi. \end{cases} \tag{20}$$

Since, the solution (20) of (19) also satisfied the following conditions

$$\frac{\partial^i \Re_1(\varphi,\varphi)}{\partial \varrho^i} = \frac{\partial^i \Re_2(\varphi,\varphi)}{\partial \varrho^i}, \quad 0 \le i \le 2m-2, \tag{21}$$

$$\frac{\partial^{2m-1} \Re_1(\varphi^+,\varphi)}{\partial \varrho^{2m-1}} - \frac{\partial^{2m-1} \Re_2(\varphi^-,\varphi)}{\partial \varrho^{2m-1}} = \frac{1}{(-1)^m}. \tag{22}$$

Using boundary conditions (17), (18), (21) and (22), we can derive the reproducing kernel $\Re(\varrho,\varphi)$ for any m.

5. Reproducing Kernel for Periodic Boundary Conditions

In this section, we will derive reproducing kernel for $m = 2$ with periodic boundary conditions.

5.1. Reproducing Kernel $\Re(\varrho,\varphi)$ for $f(\alpha) = f(\beta)$

The function space $\mathcal{W}_2^2[\alpha,\beta]$ is defined as, $\mathcal{W}_2^2[\alpha,\beta] = \{f(\varrho) : f(\varrho), f'(\varrho)$ are absolutely continuous, $f''(\varrho) \in L^2[\alpha,\beta], \varrho \in [\alpha,\beta], f(\alpha) = f(\beta)\}$. Described method in Section 4 with the periodic boundary conditions, we get fourth order differential equation with the boundary conditions in the form

$$\frac{\partial^4 \Re(\varrho,\varphi)}{\partial \varrho^4} = 0. \tag{23}$$

$$\begin{aligned}
&\Re(\alpha,\varphi) + \frac{\partial^3 \Re(\alpha,\varphi)}{\partial \varrho^3} + R(\beta,\varphi) - \frac{\partial^3 \Re(\beta,\varphi)}{\partial \varrho^3} = 0,\\
&\frac{\partial \Re(\alpha,\varphi)}{\partial \varrho} - \frac{\partial^2 \Re(\alpha,\varphi)}{\partial \varrho^2} = 0,\\
&\Re(\alpha,\varphi) - \Re(\beta,\varphi) = 0,\\
&\frac{\partial^2 \Re(\beta,\varphi)}{\partial \varrho^2} + \frac{\partial \Re(\beta,\varphi)}{\partial \varrho} = 0,\\
&\Re_1(\varphi,\varphi) = \Re_2(\varphi,\varphi),\\
&\frac{\partial \Re_1(\varphi,\varphi)}{\partial \varrho} = \frac{\partial \Re_2(\varphi,\varphi)}{\partial \varrho},\\
&\frac{\partial^2 \Re_1(\varphi,\varphi)}{\partial \varrho^2} = \frac{\partial^2 \Re_2(\varphi,\varphi)}{\partial \varrho^2},\\
&\frac{\partial^3 \Re_1(\varphi^+,\varphi)}{\partial \varrho^3} - \frac{\partial^3 \Re_2(\varphi^-,\varphi)}{\partial \varrho^3} = 1.
\end{aligned} \tag{24}$$

Hence, the solution of (23) and (24) is

$$\Re(\varrho,\varphi) = \begin{cases} \Re_1(\varrho,\varphi) = c_1(\varphi) + c_2(\varphi)\varrho + c_3(\varphi)\varrho^2 + c_4(\varphi)\varrho^3, & \varrho \le \varphi \\ \Re_2(\varrho,\varphi) = d_1(\varphi) + d_2(\varphi)\varrho + d_3(\varphi)\varrho^2 + d_4(\varphi)\varrho^3, & \varrho > \varphi. \end{cases} \tag{25}$$

Where,

$$c_1(\varphi) = \beta^2 - \frac{\beta^2\varphi}{2} - \beta\varphi + \frac{\beta^3}{3} + \frac{\varphi^3}{6}$$
$$- \frac{\beta\begin{pmatrix}\beta^5 - 3\beta^4\varphi + 18\beta^4 + 3\beta^3\varphi^2 - 45\beta^3\varphi + 36\beta^3\\ -\beta^2\varphi^3 + 36\beta^2\varphi^2 - 72\beta^2\varphi - 9\beta\varphi^3 + 54\beta\varphi^2 - 18\varphi^3\end{pmatrix}}{108\,(\alpha-\beta)}$$
$$- \frac{\beta^3(\beta-\varphi)^3}{18(\alpha-\beta)^2} - \frac{\beta\,(\beta+2)\,\left(\beta^2 - 2\beta\varphi + 2\beta + \varphi^2 - 2\varphi\right)}{4\,(\beta-\alpha+2)}$$
$$- \frac{\beta\left(\beta^2 + 9\beta + 18\right)\begin{pmatrix}\beta^3 - 3\beta^2\varphi + 9\beta^2 + 3\beta\varphi^2\\ -18\beta\varphi + 18\beta - \varphi^3 + 9\varphi^2 - 18\varphi\end{pmatrix}}{108\,(\beta-\alpha+6)} + \frac{1}{2},$$

$$c_2(\varphi) = -\frac{(\beta-\varphi)\begin{pmatrix}\alpha^4\beta - \alpha^4\varphi + 2\alpha^4 - 4\alpha^3\beta + 4\alpha^3\varphi\\ -8\alpha^3 - \alpha^2\beta^3 - \alpha^2\beta^2\varphi - 4\alpha^2\beta^2\\ +2\alpha^2\beta\varphi^2 - 4\alpha^2\beta\varphi + 2\alpha^2\varphi^2 + 2\alpha\beta^3\varphi\end{pmatrix}}{2(\alpha-\beta)^2\,(\beta-\alpha+2)\,(\beta-\alpha+6)}$$
$$- \frac{(\beta-\varphi)\begin{pmatrix}2\alpha\beta^3 - 2\alpha\beta^2\varphi^2 + 10\alpha\beta^2\varphi + 8\alpha\beta^2\\ -8\alpha\beta\varphi^2 + 8\alpha\beta\varphi - 4\alpha\varphi^2 - 2\beta^3\varphi\\ +2\beta^2\varphi^2 - 8\beta^2\varphi + 4\beta\varphi^2\end{pmatrix}}{2(\alpha-\beta)^2\,(\beta-\alpha+2)\,(\beta-\alpha+6)},$$

$$c_3(\varphi) = -\frac{-\frac{(\beta-\varphi)(2\beta-2\varphi+4)\alpha^3}{2} + \frac{(\beta-\varphi)\left(2\beta^2-\beta\varphi+12\beta-\varphi^2-6\varphi+12\right)\alpha^2}{2}}{(\alpha-\beta)^2\,(\beta-\alpha+2)\,(\beta-\alpha+6)}$$
$$- \frac{\frac{(\beta-\varphi)\left(4\beta^2-2\beta\varphi+12\beta-2\varphi^2\right)\alpha}{2}}{(\alpha-\beta)^2\,(\beta-\alpha+2)\,(\beta-\alpha+6)}$$
$$- \frac{\frac{(\beta-\varphi)\left(-\beta^3\varphi+\beta^2\varphi^2-4\beta^2\varphi+2\beta\varphi^2\right)}{2}}{(\alpha-\beta)^2\,(\beta-\alpha+2)\,(\beta-\alpha+6)},$$

$$c_4(\varphi) = -\frac{\frac{(\beta-\varphi)\left(\beta^2+\beta\varphi+6\beta-2\varphi^2\right)}{6} - \frac{\alpha(\beta-\varphi)(3\beta-3\varphi+6)}{6}}{(\alpha-\beta)^2\,(\beta-\alpha+6)},$$

$$d_1(\varphi) = \beta^2 - \frac{\beta^2\varphi}{2} - \beta\varphi + \frac{\beta^3}{3}$$
$$- \frac{\beta\begin{pmatrix}\beta^5 - 3\beta^4\varphi + 18\beta^4 + 3\beta^3 y^2 - 45\beta^3\varphi + 36\beta^3\\ -\beta^2\varphi^3 + 36\beta^2\varphi^2 - 72\beta^2\varphi - 9\beta\varphi^3 + 54\beta\varphi^2 - 18\varphi^3\end{pmatrix}}{108\,(\alpha-\beta)}$$

$$
-\frac{\beta^3 (\beta-\varphi)^3}{18(\alpha-\beta)^2}-\frac{\beta\ (\beta+2)\ \left(\beta^2-2\,\beta\,\varphi+2\,\beta+\varphi^2-2\,\varphi\right)}{4\ (\beta-\alpha+2)}
$$

$$
-\frac{\beta\ \left(\beta^2+9\,\beta+18\right)\ \begin{pmatrix}\beta^3-3\,\beta^2\,\varphi+9\,\beta^2+3\,\beta\,\varphi^2-18\,\beta\,\varphi\\ +18\,\beta-\varphi^3+9\,\varphi^2-18\,\varphi\end{pmatrix}}{108\ (\beta-\alpha+6)}
$$

$$
+\frac{1}{2},
$$

$$
d_2(\varphi) = -\frac{(\alpha-\varphi)\begin{pmatrix}\alpha^3\,\beta^2-2\,\alpha^3\,\beta\,\varphi+2\,\alpha^3\,\beta-2\,\alpha^3\,\varphi\\ +\alpha^2\,\beta^2\,\varphi-4\,\alpha^2\,\beta^2+2\,\alpha^2\,\beta\,\varphi^2+10\,\alpha^2\,\beta\,\varphi\\ -8\,\alpha^2\,\beta+2\,\alpha^2\,\varphi^2+8\,\alpha^2\,\varphi\end{pmatrix}}{2\,(\alpha-\beta)^2\ (\beta-\alpha+2)\ (\beta-\alpha+6)}
$$

$$
-\frac{(\alpha-\varphi)\begin{pmatrix}-\alpha\,\beta^4-4\,\alpha\,\beta^3-2\,\alpha\,\beta^2\,\varphi^2-4\,\alpha\,\beta^2\,\varphi\\ -8\,\alpha\,\beta\,\varphi^2-8\,\alpha\,\beta\,\varphi-4\,\alpha\,\varphi^2+\beta^4\,\varphi\\ +2\,\beta^4+4\,\beta^3\,\varphi+8\,\beta^3+2\,\beta^2\,\varphi^2+4\,\beta\,\varphi^2\end{pmatrix}}{2\,(\alpha-\beta)^2\ (\beta-\alpha+2)\ (\beta-\alpha+6)},
$$

$$
d_3(\varphi) = -\frac{-\frac{(\alpha-\varphi)\,(2\,\varphi-2\,\alpha+4)\,\beta^3}{2}+\frac{(\alpha-\varphi)\,\left(-2\,\alpha^2+\alpha\,\varphi+12\,\alpha+\varphi^2-6\,\varphi-12\right)\beta^2}{2}}{(\alpha-\beta)^2\ (\beta-\alpha+2)\ (\beta-\alpha+6)}
$$

$$
-\frac{\frac{(\alpha-\varphi)\,\left(\alpha^3\,\varphi-\alpha^2\,\varphi^2-4\,\alpha^2\,\varphi+2\,\alpha\,\varphi^2\right)}{2}+\frac{(\alpha-\varphi)\,\left(-4\,\alpha^2+2\,\alpha\,\varphi+12\,\alpha+2\,\varphi^2\right)\beta}{2}}{(\alpha-\beta)^2\ (\beta-\alpha+2)\ (\beta-\alpha+6)},
$$

$$
d_4(\varphi) = \frac{\frac{(\alpha-\varphi)\,\left(-\alpha^2-\alpha\,\varphi+6\,\alpha+2\,\varphi^2\right)}{6}-\frac{\beta\,(\alpha-\varphi)\,(3\,\varphi-3\,\alpha+6)}{6}}{(\alpha-\beta)^2\ (\beta-\alpha+6)}.
$$

5.2. Reproducing Kernel $\Re(\varrho,\varphi)$ for $f'(\alpha)=f'(\beta)$

$\mathcal{W}_2^2[\alpha,\beta] = \{f(\varrho)|f(\varrho), f'(\varrho)$ are absolutely continuous, $f''(\varrho) \in L^2[\alpha,\beta], \varrho \in [\alpha,\beta], f'(\alpha)=f'(\beta)\}$. Described method in Section 4 with the periodic boundary conditions, we get fourth order differential equation with the boundary conditions in the form

$$
\begin{aligned}
&\frac{\partial^4\Re(\varrho,\varphi)}{\partial\varrho^4}=0, \qquad (26)\\
&\Re(\alpha,\varphi)+\frac{\partial^3\Re(\alpha,\varphi)}{\partial\varrho^3}=0,\\
&\frac{\partial\Re(\alpha,\varphi)}{\partial\varrho}-\frac{\partial^2\Re(\alpha,\varphi)}{\partial\varrho^2}+\frac{\partial^2\Re(\beta,\varphi)}{\partial\varrho^2}+\frac{\partial\Re(\beta,\varphi)}{\partial\varrho}=0,
\end{aligned}
$$

$$\begin{aligned}
&\frac{\partial^3\Re(\beta,\varphi)}{\partial\varrho^3}-\Re(\beta,\varphi)=0,\\
&\frac{\partial\Re(\alpha,\varphi)}{\partial\varrho}-\frac{\partial\Re(\beta,\varphi)}{\partial\varrho}=0,\\
&\Re_1(\varphi,\varphi)=\Re_2(\varphi,\varphi),\\
&\frac{\partial\Re_1(\varphi,\varphi)}{\partial\varrho}=\frac{\partial\Re_2(\varphi,\varphi)}{\partial\varrho},\\
&\frac{\partial^2\Re_1(\varphi,\varphi)}{\partial\varrho^2}=\frac{\partial^2\Re_2(\varphi,\varphi)}{\partial\varrho^2},\\
&\frac{\partial^3\Re_1(\varphi^+,\varphi)}{\partial\varrho^3}-\frac{\partial^3\Re_2(\varphi^-,\varphi)}{\partial\varrho^3}=1.
\end{aligned}\tag{27}$$

Hence, the solution of (26) and (27) is

$$\Re(\varrho,\varphi)=\begin{cases}\Re_1(\varrho,\varphi)=c_1(\varphi)+c_2(\varphi)\varrho+c_3(\varphi)\varrho^2+c_4(\varphi)\varrho^3, & \varrho\le\varphi\\ \Re_2(\varrho,\varphi)=d_1(\varphi)+d_2(\varphi)\varrho+d_3(\varphi)\varrho^2+d_4(\varphi)\varrho^3, & \varrho>\varphi.\end{cases}\tag{28}$$

Where,

$$\begin{aligned}
c_1(\varphi)=\;&\frac{\beta^2}{2}-\frac{\beta^2\,\varphi}{2}-\frac{\beta\,\varphi}{2}+\frac{\beta^3}{3}+\frac{\varphi^3}{6}\\
&-\frac{\begin{pmatrix}6\,\beta^2-3\,\alpha\,\beta^2\\-6\,\alpha\,\beta+\beta^3+12\end{pmatrix}\begin{pmatrix}6\,\alpha\,\varphi-6\,\alpha\,\beta-6\,\beta\,\varphi\\-3\,\alpha\,\beta^2-3\,\alpha\,\varphi^2-3\,\beta\,\varphi^2\\+6\,\beta^2+\beta^3+2\,\varphi^3+6\,\alpha\,\beta\,\varphi+12\end{pmatrix}}{12\,\left(\alpha^2\,\left(3\,\beta+6\right)-\alpha\,\left(3\,\beta^2+12\,\beta\right)-\alpha^3+6\,\beta^2+\beta^3+24\right)}\\
&+\frac{\beta^2\,\left(\beta-y\right)^2}{4\,\left(\alpha-\beta\right)}+1,
\end{aligned}$$

$$\begin{aligned}
c_2(\varphi)=\;&\frac{\varphi}{2}-\frac{\alpha}{2}-\alpha\,\varphi+\frac{\alpha^2}{2}-\frac{\alpha\,\left(\alpha-\varphi\right)^2}{2\,\left(\alpha-\beta\right)}\\
&-\frac{\left(\alpha-\beta+\alpha\,\beta\right)\begin{pmatrix}6\,\beta\,\varphi-6\,\alpha\,\varphi-6\,\alpha\,\beta+3\,\alpha^2\,\beta+3\,\alpha\,\varphi^2\\+3\,\beta\,\varphi^2+6\,\alpha^2-\alpha^3-2\,\varphi^3-6\,\alpha\,\beta\,\varphi+12\end{pmatrix}}{2\,\left(\beta\,\left(12\,\alpha-3\,\alpha^2\right)+\beta^2\,\left(3\,\alpha-6\right)-6\,\alpha^2+\alpha^3-\beta^3-24\right)},
\end{aligned}$$

$$\begin{aligned}
c_3(\varphi)=\;&\frac{\left(\beta-\varphi\right)^2}{4\,\left(\alpha-\beta\right)}\\
&+\frac{\left(\alpha+\beta\right)\begin{pmatrix}6\,\alpha\,\varphi-6\,\alpha\,\beta-6\,\beta\,\varphi-3\,\alpha\,\beta^2-3\,\alpha\,\varphi^2\\-3\,\beta\,\varphi^2+6\,\beta^2+\beta^3+2\,\varphi^3+6\,\alpha\,\beta\,\varphi+12\end{pmatrix}}{4\,\left(\alpha^2\,\left(3\,\beta+6\right)-\alpha\,\left(3\,\beta^2+12\,\beta\right)-\alpha^3+6\,\beta^2+\beta^3+24\right)},
\end{aligned}$$

$$c_4(\varphi)=-\frac{\beta^2-\frac{\beta\,\varphi^2}{2}-\beta\,\varphi+\frac{\beta^3}{6}+\frac{\varphi^3}{3}-\alpha\,\left(\frac{\beta^2}{2}-\beta\,\varphi+\beta+\frac{\varphi^2}{2}-\varphi\right)+2}{\alpha^2\,\left(3\,\beta+6\right)-\alpha\,\left(3\,\beta^2+12\,\beta\right)-\alpha^3+6\,\beta^2+\beta^3+24},$$

$$d_1(\varphi) = \frac{\beta^2}{2} - \frac{\beta^2\,\varphi}{2} - \frac{\beta\,\varphi}{2} + \frac{\beta^3}{3} - \frac{\begin{pmatrix} 6\,\beta^2 - 3\,\alpha\,\beta^2 \\ -6\,\alpha\,\beta + \beta^3 + 12 \end{pmatrix} \begin{pmatrix} 6\,\alpha\,\varphi - 6\,\alpha\,\beta - 6\,\beta\,\varphi \\ -3\,\alpha\,\beta^2 - 3\,\alpha\,\varphi^2 - 3\,\beta\,\varphi^2 \\ +6\,\beta^2 + \beta^3 + 2\,\varphi^3 + 6\,\alpha\,\beta\,\varphi + 12 \end{pmatrix}}{12\,\left(\alpha^2\,\left(3\,\beta + 6\right) - \alpha\,\left(3\,\beta^2 + 12\,\beta\right) - \alpha^3 + 6\,\beta^2 + \beta^3 + 24\right)} + \frac{\beta^2\,(\beta - \varphi)^2}{4\,(\alpha - \beta)} + 1,$$

$$d_2(\varphi) = \frac{\varphi}{2} - \frac{\beta}{2} + \beta\,\varphi - \frac{\beta^2}{2} - \frac{(a - \beta + \alpha\,\beta) \begin{pmatrix} 6\,\alpha\,\varphi - 6\,\alpha\,\beta - 6\,\beta\,\varphi - 3\,\alpha\,\beta^2 - 3\,\alpha\,\varphi^2 \\ -3\,\beta\,\varphi^2 + 6\,\beta^2 + \beta^3 + 2\,\varphi^3 + 6\,\alpha\,\beta\,\varphi + 12 \end{pmatrix}}{2\,\left(\alpha^2\,\left(3\,\beta + 6\right) - \alpha\,\left(3\,\beta^2 + 12\,\beta\right) - \alpha^3 + 6\,\beta^2 + \beta^3 + 24\right)} - \frac{\beta\,(\beta - \varphi)^2}{2\,(\alpha - \beta)},$$

$$d_3(\varphi) = \frac{(\beta - \varphi)^2}{4\,(\alpha - \beta)} - \frac{\varphi}{2} + \frac{(\alpha + \beta) \begin{pmatrix} 6\,\alpha\,\varphi - 6\,\alpha\,\beta - 6\,\beta\,\varphi - 3\,\alpha\,\beta^2 - 3\,\alpha\,\varphi^2 \\ -3\,\beta\,\varphi^2 + 6\,\beta^2 + \beta^3 + 2\,\varphi^3 + 6\,\alpha\,\beta\,\varphi + 12 \end{pmatrix}}{4\,\left(\alpha^2\,\left(3\,\beta + 6\right) - \alpha\,\left(3\,\beta^2 + 12\,\beta\right) - \alpha^3 + 6\,\beta^2 + \beta^3 + 24\right)},$$

$$d_4(\varphi) = -\frac{\frac{\alpha\,\varphi^2}{2} - \alpha\,\varphi + \alpha^2 - \frac{\alpha^3}{6} - \frac{\varphi^3}{3} + \beta\,\left(\frac{\alpha^2}{2} - \alpha\,\varphi - \alpha + \frac{\varphi^2}{2} + \varphi\right) + 2}{\beta\,\left(12\,\alpha - 3\,\alpha^2\right) + \beta^2\,\left(3\,\alpha - 6\right) - 6\,\alpha^2 + \alpha^3 - \beta^3 - 24}.$$

6. Conclusion

In this paper we derived a generalized reproducing kernel for periodic boundary conditions using an improved inner product. This reproducing kernel is used to solve the first order ordinary differential equations with periodic boundary condition in particular for $m = 2$. The derive reproducing kernel is generalized to n^{th} order ordinary differential equations for substitute $m = n + 1$.

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