

Inclusion Relations of Various Subclasses of Harmonic Univalent Mappings and k -Uniformly Harmonic Starlike Functions

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Received 19 April 2021

Accepted 1 December 2021

Communicated by Peichu Hu

AMS Mathematics Subject Classification(2020): 31A05, 30C45, 33B15, 33E20.

Abstract. The purpose of the present paper is to obtain inclusion relations between various subclasses of harmonic univalent mappings by applying a convolution operator involving generalized Wright functions. To be more precise, we investigate such connections with Goodman-Rønning-type harmonic univalent functions, k -uniformly harmonic convex functions and k -uniformly harmonic starlike functions in the open unit disc \mathbb{U} . Some of our results generalize and correct the results of Maharana and Sahoo [11].

Keywords: Analytic; Harmonic functions; Harmonic convex functions; Harmonic starlike functions; Wright function.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply-connected complex domain $\mathbb{D} \subset \mathbb{C}$, if both u and v are real and harmonic in \mathbb{D} . In any simply-connected domain we can be uniquely expressed as $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} , called the analytic and co-analytic part of

the function f , respectively. Let \mathcal{H} denote the class of functions of the form $f = h + \bar{g}$, which are harmonic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the condition $f(0) = f_z(0) - 1 = 0$.

If $f = h + \bar{g} \in \mathcal{H}$, then h and g can be expressed in Taylor series expansion of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (1)$$

Further, we denote $\mathcal{H}^0 \subseteq \mathcal{H}$ consisting of functions of the form (1) with $b_1 = 0$. Let $\mathcal{S}_{\mathcal{H}}$ represent the class of all harmonic functions $f = h + \bar{g}$ of the form (1) which are univalent and sense-preserving in the open unit disk \mathbb{U} . We further let $\mathcal{S}_{\mathcal{H}}^0$ represent the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of functions of the form (1) with $b_1 = 0$. In other words, we say that $\mathcal{S}_{\mathcal{H}}^0 \equiv \mathcal{S}_{\mathcal{H}} \cap \mathcal{H}^0$.

The family $\mathcal{S}_{\mathcal{H}}^0$ is compact and normal while the class $\mathcal{S}_{\mathcal{H}}$ is normal only but not compact. For detailed study, one may refer the excellent text book by Duren [7] or Ponnusamy and Rasila [13] (see also [4, 6, 12, 15, 16]). The geometric subclasses of $\mathcal{S}_{\mathcal{H}}$ consisting of starlike, convex and close-to-convex functions in \mathbb{D} are denoted by $\mathcal{S}_{\mathcal{H}}^*$, $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}$, respectively and $\mathcal{S}_{\mathcal{H}}^{*,0} = \mathcal{S}_{\mathcal{H}}^* \cap \mathcal{H}^0$, $\mathcal{K}_{\mathcal{H}}^{*,0} = \mathcal{K}_{\mathcal{H}}^* \cap \mathcal{H}^0$, $\mathcal{C}_{\mathcal{H}}^{*,0} = \mathcal{C}_{\mathcal{H}}^* \cap \mathcal{H}^0$.

A function f of the form (1) is said to be in the class $\mathcal{N}_{\mathcal{H}}(\beta)$, if it satisfy the condition

$$\Re \left(\frac{f'(z)}{z'} \right) \geq \beta, \quad 0 \leq \beta < 1.$$

A function f of the form (1) is said to be in the class $\mathcal{G}_{\mathcal{H}}(\beta)$, if it satisfy the condition

$$\Re \left((1 + e^{i\alpha}) \frac{z f'(z)}{f(z)} - e^{i\alpha} \right) \geq \beta, \quad \alpha \in \mathbb{R}, \quad 0 \leq \beta < 1,$$

where $z' = \frac{\partial}{\partial \theta}(z \Rightarrow r e^{i\theta})$ and $f'(z) = \frac{\partial}{\partial \theta} f(r e^{i\theta})$. Further, we suppose that \mathcal{T} consist of the functions $f = h + \bar{g}$ in \mathcal{H} so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \quad (2)$$

The class \mathcal{T} was initially introduced and studied by Jahangiri [9] (see also [20, 21]). The classes $\mathcal{N}_{\mathcal{H}}(\beta)$, $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{G}_{\mathcal{H}}(\beta)$, $\mathcal{TG}_{\mathcal{H}}(\beta)$, were studied earlier by Ahuja et al. [2] and Rosy et al. [19]. The applications of hypergeometric function, generalized hypergeometric function, Wright function, Mittag-Leffler function, generalized Wright hypergeometric function on univalent functions are interesting topic of research in geometric function theory. Several researchers obtain various fruitful results by applying these functions. Noteworthy contribution in this direction may be found in [1, 11, 14, 15, 16, 17, 18, 22, 23].

In the present paper, we give a nice application of Wright hypergeometric function on certain classes of harmonic univalent functions.

Now, we recall the definition of Wright function

$$\mathcal{W}_{\lambda, \mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}. \tag{3}$$

This function was introduced by British mathematician Wright [24] in 1933. In [24], it is proved that this is an entire function for $\lambda > -1$. Recently, Sahed and Salem [8] introduced generalized Wright function $\mathcal{W}_{\lambda, \mu}^{\gamma, \delta}(z)$ which is defined as

$$\mathcal{W}_{\lambda, \mu}^{\gamma, \delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \tag{4}$$

where $\lambda > -1, \gamma, \delta, \mu \in \mathbb{C}$, $(\gamma)_n$ is a Pochhammer symbol and defined as

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} := \begin{cases} 1 & \text{if } n = 0, \\ \gamma(\gamma + 1) \cdots (\gamma + n - 1) & \text{if } n \in \mathbb{N} \end{cases} \tag{5}$$

and symbol Γ is the Gamma function. The function $\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(z)$ is an entire function of order $\frac{1}{1+\lambda}$.

Now, we define normalized generalized Wright function in the following way

$$\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(z) = \Gamma(\mu)(z) \mathcal{W}_{\lambda, \mu}^{\gamma, \delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}. \tag{6}$$

The study of normalized generalized Wright function is of special interest because by specific values of parameters it reduces to various known special functions. Some particular cases are given below:

- (i) If we take $\gamma = \delta$, then $\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(z)$ reduces to normalized Wright function $\mathbb{W}_{\lambda, \mu}(z)$.
- (ii) If we take $\lambda = 0$, then $\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(z)$ reduces to $z \mathbb{W}(\gamma; \delta; z)$, where $\mathbb{W}(\gamma; \delta; z)$ represent the confluent hypergeometric function.
- (iii) If we take $\delta = \gamma, \lambda = 1, \mu = 1 - \nu$, then $\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(-z)$ reduces to normalized Bessel function of first kind $\mathcal{J}_{\nu}(z)$.

For complex parameters $\gamma_1, \gamma_2, \delta_1, \delta_2, \lambda_1, \lambda_2, \mu_1, \mu_2$ with $\lambda_1, \lambda_2 > -1$ and $\mu_1, \mu_2 > 0$, we get

$$\phi_1(z) = \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(z) \quad \text{and} \quad \phi_2(z) = \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(z).$$

Corresponding to the functions ϕ_1 and ϕ_2 , we consider the convolution operator

$$\Omega \left[\begin{matrix} \gamma_1, \delta_1, \lambda_1, \mu_1 \\ \gamma_2, \delta_2, \lambda_2, \mu_2 \end{matrix} \right] : \mathcal{H} \longrightarrow \mathcal{H}$$

defined as

$$\Omega \left[\begin{matrix} \gamma_1, \delta_1, \lambda_1, \mu_1 \\ \gamma_2, \delta_2, \lambda_2, \mu_2 \end{matrix} \right] f = f * (\phi_1 + \overline{\phi_2}) = h * \phi_1 + \overline{g * \phi_2}$$

for any function $f = h + \bar{g}$ in \mathcal{H} , where the symbol $'*'$ denotes the usual convolution of two analytic functions. If we write

$$H(z) + \overline{G(z)} = h * \phi_1 + \overline{g * \phi_2} \quad (7)$$

then it follows from the definition of convolution that

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma_1)_{(n-1)}}{(\delta_1)_{(n-1)}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{a_n z^n}{(n-1)!}, \quad (8)$$

$$G(z) = \sum_{n=1}^{\infty} \frac{(\gamma_2)_{(n-1)}}{(\delta_2)_{(n-1)}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{b_n z^n}{(n-1)!}. \quad (9)$$

A function f of the form (1) is said to be in the class $\mathcal{HUK}(\kappa, \beta)$ if it satisfy the condition

$$\Re \left(1 + (1 + \kappa e^{i\eta}) \frac{z^2 h''(z) + 2z g'(z) + z^2 g''(z)}{z h'(z) - z \overline{g'(z)}} \right) \geq \beta,$$

where $0 \leq k < \infty$, $0 \leq \beta < 1$ and $z \in \mathbb{U}$.

Further, we let $\mathcal{THUK}(\kappa, \beta) = \mathcal{HUK}(\kappa, \beta) \cap \mathcal{T}$. A function $f \in \mathcal{HUK}(\kappa, \beta)$ is called harmonic κ -uniformly convex functions in \mathbb{D} . The classes $\mathcal{HUK}(\kappa, \beta)$ and $\mathcal{THUK}(\kappa, \beta)$ were extensively studied by Kim et al. [10].

Analogous to class $\mathcal{HUK}(\kappa, \beta)$, Ahuja et al. [2] define the class $\mathcal{HUS}^*(\kappa, \beta)$ in the following way:

A function f of the form (1) is said to be in the class $\mathcal{HUS}^*(\kappa, \beta)$, if it satisfy the condition

$$\Re \left(\frac{z f'(z)}{z' f(z)} - \beta \right) \geq \kappa \left| \frac{z f'(z)}{z' f(z)} - 1 \right|, \quad z \in \mathbb{U},$$

where $0 \leq \kappa < \infty$ and $0 \leq \beta < 1$. Further, we let $\mathcal{THUS}^*(\kappa, \beta) = \mathcal{HUS}^*(\kappa, \beta) \cap \mathcal{T}$.

For simplicity, throughout this article we will use the notation

$$\Omega(f) := \Omega \left[\begin{matrix} \gamma_1, \delta_1, \lambda_1, \mu_1 \\ \gamma_2, \delta_2, \lambda_2, \mu_2 \end{matrix} \right] f$$

and call this the convolution image of f .

In this paper, we obtain some inclusion relations among the classes $\mathcal{N}_{\mathcal{H}}(\beta)$, $\mathcal{G}_{\mathcal{H}}(\beta)$, $\mathcal{HUK}(\kappa, \beta)$, $\mathcal{HUS}^*(\kappa, \beta)$, $\mathcal{K}_{\mathcal{H}}^0$, $\mathcal{S}_{\mathcal{H}}^{*,0}$ and $\mathcal{C}_{\mathcal{H}}^0$ by applying the convolution operator Ω over the generalized Wright function.

2. Preliminary Results

In order to prove our main results, we shall require the following results, due to Clunie and Shal-Small [5] (see also [7]), Rosy et al. [19], Kim et al. [10] and Ahuja et al. [2].

Lemma 2.1. [5] *Let $f = h + \bar{g} \in \mathcal{K}_{\mathcal{H}}^0$, where h and g are given by (1) with $b_1 = 0$. Then, for $n \geq 2$ we have*

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}.$$

Lemma 2.2. [5] *Let $f = h + \bar{g} \in \mathcal{C}_{\mathcal{H}}^0 \cup \mathcal{S}_{\mathcal{H}}^{*,0}$, where $h(z)$ and $g(z)$ are of the form (1) with $b_1 = 0$. Then, for $n \geq 2$ we have*

$$|a_n| \leq \frac{(2n+1)(n+1)}{6} \quad \text{and} \quad |b_n| \leq \frac{(2n-1)(n-1)}{6}.$$

Lemma 2.3. [19] *Let $f = h + \bar{g}$, where h and g are given by (1). If for some β , $0 \leq \beta < 1$ the inequality*

$$\sum_{n=2}^{\infty} (2n-1-\beta)|a_n| + \sum_{n=1}^{\infty} (2n+1+\beta)|b_n| \leq 1-\beta \tag{10}$$

is satisfied, then f is a sense-preserving harmonic univalent mapping lying in $\mathcal{G}_{\mathcal{H}}(\beta)$.

Remark 2.4. [19] Let $f = h + \bar{g}$ be given by (2) is in the family $\mathcal{TG}_{\mathcal{H}}(\beta)$, if and only if the coefficient condition (10) holds. Moreover, if $f \in \mathcal{TG}_{\mathcal{H}}(\beta)$, then

$$|a_n| \leq \frac{1-\beta}{2n-1-\beta}, \quad n \geq 2 \quad \text{and} \quad |b_n| \leq \frac{1-\beta}{2n+1+\beta}, \quad n \geq 1.$$

Lemma 2.5. [3] *Let $f = h + \bar{g}$ where h and g are given by (2) and suppose that $0 \leq \beta < 1$. Then we have*

$$f \in \mathcal{TN}_{\mathcal{H}}(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1-\beta.$$

Remark 2.6. [3] If $f = h + \bar{g} \in \mathcal{TN}_{\mathcal{H}}(\beta)$ where h and g are given by (2), then

$$|a_n| \leq \frac{1-\beta}{n} \quad \text{for } n \geq 2 \quad \text{and} \quad |b_n| \leq \frac{1-\beta}{n} \quad \text{for } n \geq 1.$$

Lemma 2.7. [10] *Let $f = h + \bar{g}$ where h and g are given by (1). If for some κ , $0 \leq \kappa < \infty$ and $\beta, 0 \leq \beta < 1$, then the inequality*

$$\sum_{n=2}^{\infty} n\{n(\kappa+1) - (\kappa+\beta)\}|a_n| + \sum_{n=1}^{\infty} n\{n(\kappa+1) + (\kappa+\beta)\}|b_n| \leq 1-\beta, \tag{11}$$

is satisfied, then f is sense preserving harmonic univalent in \mathbb{U} and $f \in \mathcal{HUK}(\kappa, \beta)$.

Remark 2.8. [10] A mapping $f = h + \bar{g}$, where h and g are given by (2), belongs to the family $\mathcal{THUK}(\kappa, \beta)$ if and only if the condition (11) holds. Moreover, if $f \in \mathcal{THUK}(\kappa, \beta)$, then the coefficient inequalities

$$|a_n| \leq \frac{1 - \beta}{n\{n(\kappa + 1) - (\kappa + \beta)\}}, \quad n \geq 2 \quad \text{and} \quad |b_n| \leq \frac{1 - \beta}{n\{n(\kappa + 1) + (\kappa + \beta)\}}$$

hold for $n \geq 1$.

Lemma 2.9. [2] Let $f = h + \bar{g}$, where h and g are given by (1). If for some κ , $0 \leq \kappa < \infty$ and β , $0 \leq \beta < 1$, the inequality

$$\sum_{n=2}^{\infty} \{n(\kappa + 1) - (\kappa + \beta)\} |a_n| + \sum_{n=1}^{\infty} \{n(\kappa + 1) + (\kappa + \beta)\} |b_n| \leq 1 - \beta, \quad (12)$$

is satisfied, then f is sense preserving harmonic univalent in \mathbb{U} and $f \in \mathcal{HUS}^*(\kappa, \beta)$.

Remark 2.10. [2] A mapping $f = h + \bar{g}$, where h and g are given by (2), belongs to the family $\mathcal{THUS}^*(\kappa, \beta)$ if and only if the condition (12) is satisfied. Moreover, if $f \in \mathcal{THUS}^*(\kappa, \beta)$, then the coefficient inequalities

$$|a_n| \leq \frac{1 - \beta}{n(\kappa + 1) - (\kappa + \beta)}, \quad n \geq 2 \quad \text{and} \quad |b_n| \leq \frac{1 - \beta}{n(\kappa + 1) + (\kappa + \beta)}, \quad n \geq 1$$

hold.

The following relations are an easy consequences of the definition of $\mathbb{W}_{\lambda, \mu}^{\gamma, \delta}$, which are useful in the proof of our main results.

Lemma 2.11. For all $\gamma \geq 0$, $\lambda \geq 0$ and $\mu \geq 0$, we have

$$\begin{aligned} \text{(i)} \quad & \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)} \frac{1}{(n+1)!} = \mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(1) - 1; \\ \text{(ii)} \quad & \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)} \frac{1}{(n)!} = \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)'(1) - \mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(1); \\ \text{(iii)} \quad & \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)} \frac{1}{(n-1)!} = \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)''(1) - 2 \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)'(1) + 2 \mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(1); \\ \text{(iv)} \quad & \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)} \frac{1}{(n-2)!} = \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)'''(1) - 3 \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)''(1) \\ & + 6 \left(\mathbb{W}_{\lambda, \mu}^{\gamma, \delta} \right)'(1) - 6 \mathbb{W}_{\lambda, \mu}^{\gamma, \delta}(1). \end{aligned}$$

Remark 2.12. The results of above lemma, generalize the results of Maharana and Sahoo [11]. It is worthy to note that for $\gamma = \delta$, our results correct the corresponding results of [11].

3. Main Results

In our first theorem, we obtain a sufficient condition for the inclusion relation between the classes $\mathcal{K}_{\mathcal{H}}^0$ and $\mathcal{G}_{\mathcal{H}}(\beta)$.

Theorem 3.1. *Let $\gamma_1, \gamma_2, \delta_1, \delta_2 \geq 0, \mu_1, \mu_2 > 0$ and $\lambda_1, \lambda_2 \geq 0$. If for some $\beta, 0 \leq \beta < 1$, the inequality*

$$\begin{aligned} & 2 \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) + (3 - \beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) - (1 + \beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)(1) \\ & + 2 \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) + (1 + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) - (1 + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)(1) \\ & \leq 4(1 - \beta), \end{aligned}$$

is satisfied, then $\Omega(\mathcal{K}_{\mathcal{H}}^0) \subset \mathcal{G}_{\mathcal{H}}(\beta)$.

Proof. Let $f = h + \bar{g} \in \mathcal{K}_{\mathcal{H}}^0$ where h and g are given by (1) with $b_1 = 0$. Here we need to prove that $\Omega(f) = H + \bar{G} \in \mathcal{G}_{\mathcal{H}}(\beta)$, for this it is sufficient to prove that $P_1 \leq 1 - \beta$, where

$$\begin{aligned} P_1 &= \sum_{n=2}^{\infty} (2n - 1 - \beta) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} |a_n| \\ &+ \sum_{n=2}^{\infty} (2n + 1 + \beta) \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} |b_n|. \end{aligned} \tag{13}$$

Applying Lemma 2.1, we have

$$\begin{aligned} P_1 &\leq \frac{1}{2} \left\{ \sum_{n=2}^{\infty} (2n - 1 - \beta)(n + 1) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (2n + 1 + \beta)(n - 1) \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right\} \\ &= \frac{1}{2} \left\{ \sum_{n=2}^{\infty} [2(n-1)(n-2) + (7-\beta)(n-1) + 2(1-\beta)] \right. \\ &\quad \times \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \\ &\quad \left. + \sum_{n=2}^{\infty} [2(n-2) + 5 + \beta] \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-2)!} \right\} \\ &= \frac{1}{2} \left\{ 2 \sum_{n=3}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-3)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (7 - \beta) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-2)!} \right\} \end{aligned}$$

$$\begin{aligned}
 & +2(1-\beta) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1)+\mu_1)} \frac{1}{(n-1)!} \\
 & +2 \sum_{n=3}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1)+\mu_2)} \frac{1}{(n-3)!} \\
 & + (5+\beta) \sum_{n=2}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1)+\mu_2)} \frac{1}{(n-2)!} \Big\} \\
 = & \frac{1}{2} \left\{ 2 \sum_{n=1}^{\infty} \frac{(\gamma_1)_{n+1}}{(\delta_1)_{n+1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n+1)+\mu_1)} \frac{1}{(n-1)!} \right. \\
 & + (7-\beta) \sum_{n=0}^{\infty} \frac{(\gamma_1)_{n+1}}{(\delta_1)_{n+1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n+1)+\mu_1)} \frac{1}{n!} \\
 & + 2(1-\beta) \sum_{n=0}^{\infty} \frac{(\gamma_1)_{n+1}}{(\delta_1)_{n+1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n+1)+\mu_1)} \frac{1}{(n+1)!} \\
 & + 2 \sum_{n=1}^{\infty} \frac{(\gamma_2)_{n+1}}{(\delta_2)_{n+1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n+1)+\mu_2)} \frac{1}{(n-1)!} \\
 & \left. + (5+\beta) \sum_{n=0}^{\infty} \frac{(\gamma_2)_{n+1}}{(\delta_2)_{n+1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n+1)+\mu_2)} \frac{1}{(n)!} \right\} \\
 = & \frac{1}{2} \left\{ 2 \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) + (3-\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) - (1+\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)(1) \right. \\
 & + 2 \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) + (1+\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) \\
 & \left. - (1+\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)(1) - 2(1-\beta) \right\} \\
 \leq & (1-\beta)
 \end{aligned}$$

by given hypothesis. This completes the proof. ■

If we put $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2. *Let $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 > 0$. If for some $\beta, 0 \leq \beta < 1$ and the inequality*

$$\begin{aligned}
 & 2\mathbb{W}_{\lambda_1, \mu_1}''(1) + (3-\beta)\mathbb{W}_{\lambda_1, \mu_1}'(1) - (1+\beta)\mathbb{W}_{\lambda_1, \mu_1}(1) \\
 & + 2\mathbb{W}_{\lambda_2, \mu_2}''(1) + (1+\beta)\mathbb{W}_{\lambda_2, \mu_2}'(1) - (1+\beta)\mathbb{W}_{\lambda_2, \mu_2}(1) \\
 & \leq 4(1-\beta),
 \end{aligned}$$

is satisfied, then the inclusion relation $\Omega(\mathcal{K}_{\mathcal{H}}^0) \subset \mathcal{G}_{\mathcal{H}}(\beta)$ holds.

Remark 3.3. It is worthy to note that the result of above corollary correct the corresponding result of Maharana and Sahoo [11].

Theorem 3.4. *Let $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$ and $\lambda_1, \lambda_2 \geq 0$. If for some $\beta, 0 \leq \beta < 1$, the inequality*

$$\begin{aligned} &4 \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'''(1) + (16 - 2\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) + (7 - 5\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) \\ &- (1 + \beta) \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) + 4 \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'''(1) + (8 + 2\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) \\ &- (1 + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) + (1 + \beta) \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \leq 12(1 - \beta), \end{aligned}$$

is satisfied, then the inclusion relations $\Omega(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{G}_{\mathcal{H}}(\beta)$ and $\Omega(\mathcal{S}_{\mathcal{H}}^{,0}) \subset \mathcal{G}_{\mathcal{H}}(\beta)$ hold.*

Proof. Let $f = h + \bar{g}$, where h and g are given by (1) with $b_1 = 0$ and $f \in \mathcal{C}_{\mathcal{H}}^0$ or $f \in \mathcal{S}_{\mathcal{H}}^{*,0}$. To prove that $\Omega(f) = H + \bar{G} \in \mathcal{G}_{\mathcal{H}}(\beta)$, where H and G are defined by (8) and (9), it is sufficient to prove that $P_1 \leq 1 - \beta$, where P_1 is given by (13). Using Lemma 2.2, we have

$$\begin{aligned} P_1 &\leq \frac{1}{6} \left\{ \sum_{n=2}^{\infty} (2n - 1 - \beta)(2n + 1)(n + 1) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 1)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (2n + 1 + \beta)(2n - 1)(n - 1) \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n - 1) + \mu_2)} \frac{1}{(n - 1)!} \right\} \\ &= \frac{1}{6} \left\{ \sum_{n=2}^{\infty} \{4(n - 1)(n - 2)(n - 3) + (28 - 2\beta)(n - 1)(n - 2) \right. \\ &\quad \left. + (39 - 9\beta)(n - 1) + 6(1 - \beta)\} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 1)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} \{4(n - 2)(n - 3) + (20 + 2\beta)(n - 2) + (15 + 3\beta)\} \right. \\ &\quad \left. \times \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n - 1) + \mu_2)} \frac{1}{(n - 2)!} \right\} \\ &= \frac{1}{6} \left\{ 4 \sum_{n=4}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 4)!} \right. \\ &\quad \left. + (28 - 2\beta) \sum_{n=3}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 3)!} \right. \\ &\quad \left. + (39 - 9\beta) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 2)!} \right. \\ &\quad \left. + 6(1 - \beta) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n - 1) + \mu_1)} \frac{1}{(n - 1)!} \right. \\ &\quad \left. + 4 \sum_{n=4}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n - 1) + \mu_2)} \frac{1}{(n - 4)!} \right\} \end{aligned}$$

$$\begin{aligned}
 & + (20 + 2\beta) \sum_{n=3}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-3)!} \\
 & + (15 + 3\beta) \sum_{n=2}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-2)!} \Big\} \\
 = & \frac{1}{6} \left\{ 4 \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'''(1) + (16 - 2\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) + (7 - 5\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) \right. \\
 & - (1 + \beta) \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) - 6(1 - \beta) + 4 \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'''(1) \\
 & + (8 + 2\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) \\
 & \left. - (1 + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) + (1 + \beta) \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \right\} \\
 \leq & (1 - \beta),
 \end{aligned}$$

by given hypothesis. Thus the proof is established. ■

Remark 3.5. It is worthy to note that for $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$ the result of Theorem 3.4 correct the corresponding result of Maharana and Sahoo [11].

Theorem 3.6. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2 > 0, \mu_1, \mu_2 > 0$. If the inequality

$$\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) + \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \leq 2. \tag{14}$$

is satisfied, then for $\beta, 0 \leq \beta < 1$, we have $\Omega(\mathcal{T}\mathcal{G}_{\mathcal{H}}(\beta) \subset \mathcal{G}_{\mathcal{H}}(\beta))$.

Proof. Let $f = h + \bar{g} \in \mathcal{T}\mathcal{G}_{\mathcal{H}}(\beta)$, where h and g are given by (2). To prove $\Omega(f) \in \mathcal{G}_{\mathcal{H}}(\beta)$, it is sufficient to prove that $P_2 \leq 1 - \beta$, where

$$\begin{aligned}
 P_2 = & \sum_{n=2}^{\infty} (2n - 1 - \beta) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} |a_n| \\
 & + \sum_{n=1}^{\infty} (2n + 1 + \beta) \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} |b_n|.
 \end{aligned} \tag{15}$$

Using Remark 2.4, we have

$$\begin{aligned}
 P_2 \leq & (1 - \beta) \left\{ \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right\} \\
 = & (1 - \beta) \left\{ \sum_{n=1}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n+1)!} \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=0}^{\infty} \frac{(\gamma_2)_n}{(\delta_2)_n} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2 n + \mu_2)} \frac{1}{n!} \Big\} \\
 & = (1 - \beta) \left\{ \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) - 1 + \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \right\} \\
 & \leq (1 - \beta),
 \end{aligned}$$

by given hypothesis. This completes the proof. ■

Lemma 3.7. *For all $\lambda \geq 0$ and $\mu > \lambda$, $\gamma, \delta > 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \frac{1}{(n+1)!} = \left(\frac{\delta - 1}{\gamma - 1} \right) \frac{\Gamma(\mu)}{\Gamma(\mu - \lambda)} \left[\mathbb{W}_{\lambda, \mu - \lambda}^{\gamma - 1, \delta - 1}(1) - 1 \right].$$

Proof.

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \frac{1}{(n+1)!} \\
 & = \left(\frac{\delta - 1}{\gamma - 1} \right) \frac{\Gamma(\mu)}{\Gamma(\mu - \lambda)} \sum_{n=0}^{\infty} \frac{(\gamma - 1)_{n+1}}{(\delta - 1)_{n+1}} \frac{\Gamma(\mu - \lambda)}{\Gamma(\lambda(n+1) + \mu - \lambda)} \frac{1}{(n+1)!} \\
 & = \left(\frac{\delta - 1}{\gamma - 1} \right) \frac{\Gamma(\mu)}{\Gamma(\mu - \lambda)} \left[\mathbb{W}_{\lambda, \mu - \lambda}^{\gamma - 1, \delta - 1}(1) - 1 \right].
 \end{aligned}$$

■

Theorem 3.8. *Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2 > 1$, $\mu_1 > \lambda_1$, $\mu_2 > \lambda_2$. If for some β , $0 \leq \beta < 1$, the inequality*

$$\begin{aligned}
 & 2\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) + 2\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) - (1 + \beta) \left\{ \left(\frac{\delta_1 - 1}{\gamma_1 - 1} \right) \frac{\Gamma(\mu_1)}{\Gamma(\mu_1 - \lambda_1)} \left[\mathbb{W}_{(\lambda_1, \mu_1 - \lambda_1)}^{\gamma_1 - 1, \delta_1 - 1}(1) - 1 \right] \right. \\
 & \left. - \left(\frac{\delta_2 - 1}{\gamma_2 - 1} \right) \frac{\Gamma(\mu_2)}{\Gamma(\mu_2 - \lambda_2)} \left[\mathbb{W}_{(\lambda_2, \mu_2 - \lambda_2)}^{\gamma_2 - 1, \delta_2 - 1}(1) - 1 \right] \right\} \leq 2 - \beta
 \end{aligned}$$

is satisfied, then $\Omega(\mathcal{TN}_{\mathcal{H}}(\beta)) \subset \mathcal{G}_{\mathcal{H}}(\beta)$.

Proof. Let $f = h + \bar{g}$ with h and g are given by (2) and $f \in \mathcal{TN}_{\mathcal{H}}(\beta)$. To prove $\Omega(f) \in \mathcal{G}_{\mathcal{H}}(\beta)$, it is sufficient to prove that $P_2 \leq 1 - \beta$, where P_2 is given by (15). Using Remark 2.6 in the definition of P_2 , we have

$$\begin{aligned}
 P_2 & \leq (1 - \beta) \left[\sum_{n=2}^{\infty} \left(2 - \frac{1 + \beta}{n} \right) \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \right. \\
 & \quad \left. + \sum_{n=1}^{\infty} \left(2 + \frac{1 + \beta}{n} \right) \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right] \\
 & = (1 - \beta) \left[2 \sum_{n=0}^{\infty} \frac{(\gamma_1)_{n+1}}{(\delta_1)_{n+1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n+1) + \mu_1)} \frac{1}{(n+1)!} \right.
 \end{aligned}$$

$$\begin{aligned}
 & - (1 + \beta) \sum_{n=1}^{\infty} \frac{(\gamma_1)_n}{(\delta_1)_n} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n) + \mu_1)} \frac{1}{(n+1)!} \\
 & + 2 \sum_{n=0}^{\infty} \frac{(\gamma_2)_n}{(\delta_2)_n} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n) + \mu_2)} \frac{1}{n!} \\
 & + (1 + \beta) \sum_{n=0}^{\infty} \frac{(\gamma_2)_n}{(\delta_2)_n} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n) + \mu_2)} \frac{1}{(n+1)!} \Big] \\
 & = (1 - \beta) \left[2 \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) - 1 \right) - (1 + \beta) \left(\sum_{n=0}^{\infty} \frac{(\gamma_1)_n}{(\delta_1)_n} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n) + \mu_1)} \frac{1}{(n+1)!} - 1 \right) \right. \\
 & \quad \left. + 2 \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) + (1 + \beta) \frac{\delta_2 - 1}{\gamma_2 - 1} \frac{\Gamma(\mu_2)}{\Gamma(\mu_2 - \lambda_2)} \left[\mathbb{W}_{\lambda_2, \mu_2 - \lambda_2}^{\gamma_2 - 1, \delta_2 - 1}(1) - 1 \right] \right] \\
 & = (1 - \beta) \left[2 \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1}(1) + 2 \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \right. \\
 & \quad \left. - (1 + \beta) \left\{ \frac{\delta_1 - 1}{\gamma_1 - 1} \frac{\Gamma(\mu_1)}{\Gamma(\mu_1 - \lambda_1)} \left[\mathbb{W}_{\lambda_1, \mu_1 - \lambda_1}^{\gamma_1 - 1, \delta_1 - 1}(1) - 1 \right] \right\} \right. \\
 & \quad \left. - \frac{\delta_2 - 1}{\gamma_2 - 1} \frac{\Gamma(\mu_2)}{\Gamma(\mu_2 - \lambda_2)} \left[\mathbb{W}_{\lambda_2, \mu_2 - \lambda_2}^{\gamma_2 - 1, \delta_2 - 1}(1) - 1 \right] - (1 - \beta) \right] \\
 & \leq 1 - \beta,
 \end{aligned}$$

by given hypothesis. Thus, the proof is established. ■

Remark 3.9. If we put $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$ in Theorems 3.6 and 3.8, then we obtain the corresponding result of Maharana and Sahoo [11].

Theorem 3.10. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$. If for some κ , $0 < \kappa < \infty$ and β , $0 \leq \beta < 1$, the inequality

$$\begin{aligned}
 & (\kappa + 1) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'''(1) + (1 - \beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) + (\kappa + 1) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'''(1) \\
 & - (1 - \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) + 2(1 - \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) - 2(1 - \beta) \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2}(1) \\
 & \leq 2(1 - \beta),
 \end{aligned}$$

is satisfied, then $\Omega(\mathcal{K}_{\mathcal{H}}^0) \subset \mathcal{HUK}(\kappa, \beta)$.

Proof. Let $f = h + \bar{g} \in \mathcal{K}_{\mathcal{H}}^0$, where h and g are given by (1) with $b_1 = 0$. Here we need to show that $\Omega(f) \in \mathcal{H} \cup \mathcal{K}(\kappa, \beta)$, for this it is sufficient to prove that $P_3 \leq 1 - \beta$, where

$$\begin{aligned}
 P_3 & = \sum_{n=2}^{\infty} n \{n(\kappa + 1) - (\kappa + \beta)\} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} |a_n| \\
 & \quad + \sum_{n=1}^{\infty} n \{n(\kappa + 1) + (\kappa + \beta)\} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} |b_n|.
 \end{aligned} \tag{16}$$

Now, applying Lemma 2.1, we have

$$\begin{aligned}
 p_3 &\leq \frac{1}{2} \left[\sum_{n=2}^{\infty} n(n+1) \{n(\kappa+1) - (\kappa+\beta)\} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} n \{n(\kappa+1) + (\kappa+\beta)\} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-2)!} \right] \\
 &= \frac{1}{2} \left[\sum_{n=2}^{\infty} ((\kappa+1)(n-1)(n-2)(n-3) + (3\kappa+4-\beta)(n-1)(n-2)) \right. \\
 &\quad + 2(1-\beta) \times \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \\
 &\quad + \sum_{n=1}^{\infty} ((\kappa+1)(n-2)(n-3) + (3\kappa+2+\beta)(n-2)) \\
 &\quad \left. \times \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right] \\
 &= \frac{1}{2} \left[(\kappa+1) \sum_{n=4}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-4)!} \right. \\
 &\quad + (3\kappa+4-\beta) \sum_{n=3}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-3)!} \\
 &\quad + 2(1-\beta) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-2)!} \\
 &\quad + (\kappa+1) \sum_{n=4}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-4)!} \\
 &\quad \left. + (3\kappa+2+\beta) \sum_{n=3}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-3)!} \right] \\
 &= \frac{1}{2} \left[(\kappa+1) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'''(1) + (1-\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) \right. \\
 &\quad + (\kappa+1) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'''(1) - (1-\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) \\
 &\quad \left. + 2(1-\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) - 2(1-\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)(1) \right] \\
 &\leq (1-\beta),
 \end{aligned}$$

by given hypothesis. Thus the proof is established. ■

In our next theorem, we obtain an inclusion relation between $\Omega\mathcal{N}_{\mathcal{H}}(\beta)$ and $\mathcal{HUK}(\kappa, \beta)$.

Theorem 3.11. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$. If for some $\kappa, 0 \leq \kappa < \infty$ and $\beta, 0 \leq \beta < 1$, the inequality

$$\begin{aligned} & (\kappa + 1) \left\{ \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)' (1) + \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)' (1) \right\} \\ & - (\kappa + \beta) \left\{ \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} (1) - \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} (1) \right\} \leq 2 - \beta, \end{aligned}$$

is satisfied, then $\Omega(\mathcal{TN}_{\mathcal{H}}(\beta) \subset \mathcal{HUK}(\kappa, \beta)$.

Proof. Let $f = h + \bar{g} \in \mathcal{TN}_{\mathcal{H}}(\beta)$, where h and g are given by (2). To prove $\Omega(f) \in \mathcal{HUK}(\kappa, \beta)$ it is sufficient to prove that $P_4 \leq 1 - \beta$, where

$$\begin{aligned} P_4 &= \sum_{n=2}^{\infty} n \{n(\kappa + 1) - (\kappa + \beta)\} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} |a_n| \\ &+ \sum_{n=1}^{\infty} n \{n(\kappa + 1) + (\kappa + \beta)\} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} |b_n|. \end{aligned} \quad (17)$$

Using Remark 2.6, we have

$$\begin{aligned} P_4 &\leq (1 - \beta) \left[\sum_{n=2}^{\infty} \{(\kappa + 1)(n-1) + (1 - \beta)\} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \{(\kappa + 1)(n-1) + (2\kappa + \beta + 1)\} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right] \\ &= (1 - \beta) \left[(\kappa + 1) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-2)!} \right. \\ &\quad + (1 - \beta) \sum_{n=2}^{\infty} \frac{(\gamma_1)_{n-1}}{(\delta_1)_{n-1}} \frac{\Gamma(\mu_1)}{\Gamma(\lambda_1(n-1) + \mu_1)} \frac{1}{(n-1)!} \\ &\quad + (\kappa + 1) \sum_{n=2}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-2)!} \\ &\quad \left. + (2\kappa + \beta + 1) \sum_{n=1}^{\infty} \frac{(\gamma_2)_{n-1}}{(\delta_2)_{n-1}} \frac{\Gamma(\mu_2)}{\Gamma(\lambda_2(n-1) + \mu_2)} \frac{1}{(n-1)!} \right] \\ &= (1 - \beta) \left[(\kappa + 1) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)' (1) - (\kappa + \beta) \mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} (1) \right. \\ &\quad \left. + (\kappa + 1) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)' (1) + (\kappa + \beta) \mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} (1) - (1 - \beta) \right] \\ &\leq 1 - \beta, \end{aligned}$$

by given hypothesis. Thus the proof is established. \blacksquare

Theorem 3.12. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$. If for some $\kappa,$

$0 \leq \kappa < \infty$ and $\beta, 0 \leq \beta < 1$, the inequality (14) holds, then $\Omega(\mathcal{THUK}(\kappa, \beta) \subset \mathcal{HUK}(\kappa, \beta)$.

Proof. The proof of the above theorem is much akin to that of Theorem 3.11. Therefore we omit the details involved. ■

Remark 3.13. If we put $\gamma_1 = \gamma_2, \delta_1 = \delta_2$ in Theorem 3.12, then we obtain the corresponding result of Maharana and Sahoo [11].

Theorem 3.14. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$. If for some $\kappa, 0 \leq \kappa < \infty$ and $\beta, 0 \leq \beta < 1$, the inequality

$$\begin{aligned}
 &(\kappa + 1) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) - (\kappa + \beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) + (\kappa + 1) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) \\
 &+ (\kappa + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) - (\kappa + \beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)(1) + (\kappa + \beta) \leq 2(1 - \beta)
 \end{aligned} \tag{18}$$

is satisfied, then $\Omega(\mathcal{K}_{\mathcal{H}}^0) \subset \mathcal{HUS}^*(\kappa, \beta)$.

Proof. The proof is much akin to that of Theorem 3.10. Hence we omit the details. ■

Theorem 3.15. Let $\lambda_1, \lambda_2 \geq 0$ and $\gamma_1, \gamma_2, \delta_1, \delta_2, \mu_1, \mu_2 > 0$. If for some $\kappa, 0 \leq \kappa < \infty$ and $\beta, 0 \leq \beta < 1$, the inequality

$$\begin{aligned}
 &2(\kappa + 1) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'''(1) + (\kappa + 3 - 2\beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)''(1) - (\kappa + \beta) \left(\mathbb{W}_{\lambda_1, \mu_1}^{\gamma_1, \delta_1} \right)'(1) \\
 &+ 2(\kappa + 1) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'''(1) - (\kappa + 3 - 2\beta) \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)''(1) \\
 &+ (\kappa + 6 - 5\beta) \left(\left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)'(1) - \left(\mathbb{W}_{\lambda_2, \mu_2}^{\gamma_2, \delta_2} \right)(1) \right) + (\kappa + \beta) \leq 6(1 - \beta)
 \end{aligned} \tag{19}$$

is satisfied, then $\Omega(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{HUS}^*(\kappa, \beta)$ or $\Omega(\mathcal{S}_{\mathcal{H}}^{*,0}) \subset \mathcal{HUS}^*(\kappa, \beta)$.

Remark 3.16. If we let $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$ in the results of Theorems 3.11, 3.12, 3.14 and 3.15, then we correct the corresponding results of Maharana and Sahoo [11].

Acknowledgement. The authors would like thank the referee for their insightful suggestions to improve the paper in the present form.

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