Southeast Asian Bulletin of Mathematics (2023) 47: 835–843

Southeast Asian Bulletin of Mathematics

On Metric Dimension of Circulant Graph $C_n(1,2)$ Joining n-paths

Sunny Kumar Sharma and Vijay Kumar Bhat
Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal, Karnataka, India
School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India
Email: sunnysrrm94@gmail.com; vijaykumarbhat2000@yahoo.com

Received 26 March 2021
Accepted 11 January 2022

Communicated by K. Denecke

AMS Mathematics Subject Classification(2020): 05C10, 05C12

Abstract. Let $H = H(V, E)$ be a graph. A subset of vertices M in $V(H)$ is said to be a resolving set (or metric generator) for H if every $y, z \in V(H)$ with $y \neq z$, there exists a vertex $a \in M$ such that $d(a, y) \neq d(a, z)$. A metric generator containing a minimum number of vertices is called a metric basis for H and the cardinality of this metric basis is the metric dimension of H, denoted by $dim(H)$. Let $C_n^q(1,2)$ be a graph obtained from the circulant graph $C_n(1,2)$ by joining n-paths of length q at each vertex of the graph $C_n(1,2)$. In this work, we show that the metric dimension of the graph $C_n^q(1,2)$ is three when $n \equiv 0, 2, 3 \ mod(4)$ and four when $n \equiv 1 \ mod(4)$.

Keywords: Circulant graph; Metric dimension; Resolving set; Pendant vertices; Pendant edges.

1. Introduction

Suppose $H = H(V, E)$ is a simple graph with E as the edge set and V as the vertex set. The distance between two vertices $y, z \in V$, denoted by $d(y, z)$, and is the length of a shortest path between y and z. The *degree* (or *valency*) of a vertex $u \in V$, denoted by d_u, is the number of edges in H containing u. If every vertex of H has a finite degree, then H is said to be a *locally finite graph.* All of the graphs considered in this work are locally finite and connected.

A vertex $z \in V$ is said to *resolve* (distinguish or recognize) two distinct vertices z_1, z_2 in H if $d(z, z_1) \neq d(z, z_2)$. Let $M = \{z_1, z_2, z_3, ..., z_p\}$ be an ordered subset of vertices and z be a vertex in H. The co-ordinate (or representation) $r(z|M)$ of z with respect to M is the p-tuple $(d(z, z_1), d(z, z_2), d(z, z_3), ..., d(z, z_p))$. Then M is said to be a *locating set* [15] or a *resolving set* [5] if distinct vertices of H have distinct co-ordinates with respect to M. A resolving set with minimum cardinality is known as the *basis* for H and this cardinality is the *metric dimension* of H, denoted by $dim(H)$.

The concepts of resolving set and metric dimension in general graphs were first introduced by Slater [15] and Harary and Melter [5]. Since then, these notions have been extensively studied. Apart from these two important initial papers [5, 15], several studies regarding applications as well as certain theoretical properties, of this invariant, are available in the literature [1, 4, 8, 9, 10, 12, 16].

Many researchers have studied the metric dimension of different graph classes. For example, the prism graph; the antiprism graph; generalized Petersen graphs $P(n, 2)$; convex polytopes (with bounded and unbounded metric dimension) [7, 13, 14]; Harary graphs $H_{4,n}$; Mobius ladders; heptagonal circular ladder [12]; circulant graphs; etc. For the last two decades, the metric dimension of circulant graphs has received a lot of attention, one can see [6, 8, 11, 17] and references therein.

In this work, we construct a graph, denoted by $C_n^q(1, 2)$, which is obtained from the circulant graph $C_n(1, 2)$ by joining n-paths of length q (≥ 1) at each vertex of the graph $C_n(1, 2)$ (see Fig. 1). In [2], the metric dimension of circulant graphs $C_n(1, 2)$ has been investigated. In this article, we extend this study to the circulant path graph $C_n^q(1, 2)$. We prove that $dim(C_n^q(1, 2)) = dim(C_n(1, 2))$, for every $n \geq 8$.

2. Preliminaries

In this section, we recall some basic definitions and results on the circulant graphs and metric dimension of graphs.

Definition 2.1. [18] *A graph H is said to be a regular graph if every vertex of H has the same degree. A graph with all of its vertices is of degree k, is called a regular graph of degree k or a k-regular graph.*

Definition 2.2. [17] *Let n, k and d_1, d_2, d_3,...,d_k be natural numbers such that $1 \leq d_1 < d_2 < d_3 < ... < d_k \leq \lfloor \frac{n}{2} \rfloor$. The circulant graph $C_n(d_1, d_2, d_3, ..., d_k)$ consists of vertices $x_{0=n}, x_1, x_2, ..., x_{n-1}$ and edges $x_l x_{l+d_p}$, where $0 \leq l \leq n-1$, $1 \leq p \leq k$, the indices are taken modulo n. The naturals d_1, d_2, d_3,...,d_k are called generators. The circulant graph $C_n(d_1, d_2, d_3, ..., d_k)$ is either a regular graph of valency $2k$ if $d_j < \frac{n}{2}$; $j = 1, 2, 3, ..., k$, or of valency $2k - 1$ if $\frac{n}{2}$ is one* *of the generator.*

By the definition of circulant graph, it is clear that $C_n(1)$ is an undirected cycle $\mathcal{C}_n$ and $C_n(1,2,...,\lfloor\frac{n}{2}\rfloor\ )$ is the complete graph K_n. Suppose $\mathcal{F}$ is a family of connected graphs H_n : $\mathcal{F} = (H_n)_{n\geq 1}$ depending upon n as follows: $\lim_{n\to\infty}\phi(n) = \infty$ and $|V(H)| = \phi(n)$. We say $\mathcal{F}$ has a bounded metric dimension if there exists a constant $D > 0$ such that $dim(H_n) \leq D$ for every $n \geq 1$; otherwise, $\mathcal{F}$ has an unbounded metric dimension. If all graphs in $\mathcal{F}$ have an equal metric dimension (i.e., independent of n), then $\mathcal{F}$ is known as the family with a constant metric dimension. Cycle graphs $\mathcal{C}_n$, path graphs P_n, heptagonal circular ladder Γ_n, prism $\mathbb{D}_n$, antiprism A_n, etc. are the families of graphs with bounded metric dimension.

Khuller et al. [9] introduced a result for those graphs having metric dimension two and is given as:

Theorem 2.3. *Let $A \subseteq V(H)$ be the metric basis for the connected graph H with cardinality two i.e., $|A| = 2$, and say $A = \{\varpi, \xi\}$. Then, the following are true:*

(i) *Between the vertices ϖ and ξ, there exists a unique shortest path P.*
(ii) *The valencies of the vertices ϖ and ξ can never exceed* 3*.*
(iii) *The valency of any other vertex on P can never exceed* 5*.*

For the circulant graphs $C_n(1,2)$, Javaid et al. [8], proved the following result:

Theorem 2.4. *For $n \geq 5$, we have*

$$dim(C_n(1,2)) \begin{cases} = 3 & if\ \ n \equiv 0,2,3\ (mod 4); \\ \leq 4 & if\ \ n \equiv 1\ (mod 4). \end{cases}$$

In [2], authors proved that $dim(C_n(1,2)) = 4$ if $n \equiv 1\ (mod4)$ and $dim(C_n(1,2)) = 3$ otherwise. In this work, we consider a family of graph $C_n^q(1,2)$ for which we have $V(C_n^q(1,2)) = \{x_j, y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq q\}$ (see Fig. 1). We denote the sets of metric co-ordinates for these vertices $x_j,\ y_j^1,\ y_j^2,\ y_j^3,\ ...,\ y_j^q$ $(1 \leq j \leq n,\ \ q \geq 1)$, respectively by $\mathbb{X}$, $\mathbb{Y}^1$, $\mathbb{Y}^2$, $\mathbb{Y}^3$, ..., $\mathbb{Y}^q$ for $C_n^q(1,2)$. We will use *resolving sets* throughout the paper rather than locating sets and all vertex indices are taken to be modulo n.

3. The Vertex Resolvability of $C_n^q(1,2)$

In this section, we study some basic properties and the metric dimension of the graph $C_n^q(1,2)$, which is obtained from the circulant graph $C_n(1,2)$.

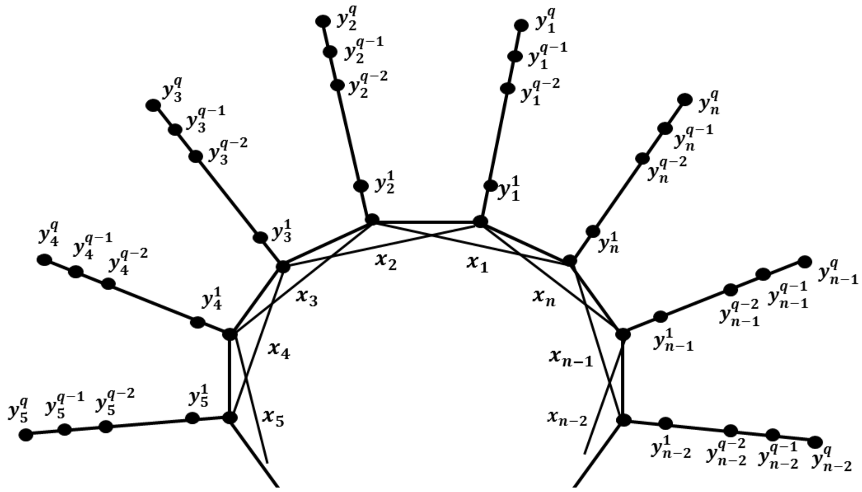

Figure 1: The graph $C_n^q(1,2)$

The graph $C_n^q(1,2)$ is obtained from the circulant graph $C_n(1,2)$ [8] by placing n new edges between the vertices of $C_n(1,2)$ and the pendant vertices of n-paths as shown in Fig. 1. The graph $C_n^q(1,2)$ has $n(q+1)$ vertices and $n(q+2)$ edges, where $q \geq 1$. The set of edges and vertices of $C_n^q(1,2)$ is depicted separately by $E(C_n^q(1,2))$ and $V(C_n^q(1,2))$, where $V(C_n^q(1,2)) = \{x_j, y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq q\}$ and $E(C_n^q(1,2)) = E(C_n(1,2)) \cup \{x_j y_j^1, y_j^l y_j^{l+1} : 1 \leq j \leq n,\ 1 \leq l \leq q-1\}$.

We call the cycle generated by vertices $\{x_j : j = 1, 2, ..., n\}$ in the graph, $C_n^q(1,2)$ as the x-cycle, and the vertices $\{y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq q\}$ as the outer vertices. In the next result, we obtain that the metric dimension of $C_n^q(1,2)$ is 3 when $n \equiv 0, 2, 3 (mod 4)$, and is 4 whenever $n \equiv 1\ (mod\ 4)$.

Theorem 3.1. *For $n \geq 8$, we have*

$$dim(C_n^q(1,2)) = \begin{cases} 3 & if\ \ n \equiv 0, 2, 3\ (mod\ 4); \\ 4 & if\ \ n \equiv 1\ (mod\ 4). \end{cases}$$

Proof. To prove this theorem, we divide our proof into the following four cases:

Case 1. $n \equiv 0\ mod(4)$.

For this, we write $n = 4w,\ w \geq 2,\ w \in \mathbb{Z}^+$. Let $\mathbb{R} = \{x_1, x_3, x_{2w+1}\} \subset \mathbb{V}(C_n^q(1,2))$. We show that $\mathbb{R}$ is a resolving set for $C_n^q(1,2)$ (for $w = 2$ it is obvious, so we take $w \geq 3$). For this, we give the co-ordinates to every element of $\mathbb{V}(C_n^q(1,2))$ with respect to $\mathbb{R}$.

The co-ordinate for the vertices $\{x_j : j = 1, 2, ..., n\}$ are

$$\gamma(x_{2k}|\mathbb{R}) = \begin{cases} (1,1,w) & k=1; \\ (k,k-1,w-k+1) & 2 \le k \le w; \\ (w,w,1) & k=w+1; \\ (2w-k+1, 2w-k+2, k-w) & w+2 \le k \le 2w \end{cases}$$

and

$$\gamma(x_{2k+1}|\mathbb{R}) = \begin{cases} (0,1,w) & k=0; \\ (1,0,w-1) & k=1; \\ (k,k-1,w-k) & 2 \le k \le w; \\ (2w-k, 2w-k+1, k-w) & w+1 \le k \le 2w-1. \end{cases}$$

The co-ordinates for the vertices $\{y_j^l : 1 \le j \le n,\ 1 \le l \le q\ \}$ are $\gamma(y_j^l|\mathbb{R}) = \gamma(x_j|\mathbb{R}) + (l,l,l)$ for $1 \le j \le n$ and $1 \le l \le q$.

From these codes, we find that $|\mathbb{X}| = |\mathbb{Y}^1| = |\mathbb{Y}^2| = ... = |\mathbb{Y}^l| = n$ and the sum of all of these cardinalities is equal to $|V(C_n^q(1,2))|$. Moreover, all of these sets are pairwise disjoint, and so we find that no pair of two distinct vertices in $C_n^q(1,2)$ are having the same metric codes, which implies that $dim(C_n^q(1,2)) \le 3$. On the other hand, we show that $dim(C_n^q(1,2)) \ge 3$ by proving that there exists no resolving set $\mathbb{R}$ such that $|\mathbb{R}| = 2$. On the contrary, suppose $dim(C_n^q(1,2)) = 2$. By Theorem 2.3, we find that the valency of basis vertices can be 0, 1, 2, or 3. But except the vertices y_j^l ($1 \le l \le n$ and $1 \le l \le q$), all other vertices of $C_n^q(1,2)$ have valency 5. Then, we have the following cases:

When the pair of vertices are in $\{y_j^l : 1 \le l \le n,\ 1 \le l \le q\}$ of the graph $C_n^q(1,2)$. Without loss of generality, we suppose that first resolving vertex is y_1^l. Suppose, second resolving vertex is y_j^l ($2 \le j \le 2w+1$ and $1 \le l \le q$). Now, again we have two cases:

Subcase 1.1. $j \equiv 0\ mod(2)$: Then for $j = 2$ and $1 \le l \le q$, we have $\gamma(y_{4w}|\{y_1^l, y_2^l\}) = \gamma(y_3|\{y_1^l, y_2^l\})$, and when $4 \le j \le 2w$ and $1 \le l \le q$, we have $\gamma(x_2|\{y_1^l, y_j^l\}) = \gamma(x_3|\{y_1^l, y_j^l\})$, a contradiction.

Subcase 1.2. $j \equiv 1\ mod(2)$: Then for $3 \le j \le 2w-1$ and $1 \le l \le q$, we have $\gamma(x_{4w}|\{y_1^l, y_j^l\}) = \gamma(x_{4w-1}|\{y_1^l, y_j^l\})$, and for $j = 2w+1$ and $1 \le l \le q$, we have $\gamma(x_2|\{y_1^l, y_j^l\}) = \gamma(x_{4w}|\{y_1^l, y_j^l\})$, a contradiction.

Hence, we find no resolving set with two vertices for $\mathbb{V}(C_n^q(1,2))$ implying that $dim(C_n^q(1,2)) = 3$ in this case.

Case 2. $n \equiv 2\ mod(4)$.

For this, we write $n = 4w+2$, $w \ge 2$, $w \in \mathbb{Z}^+$. Let $\mathbb{R} = \{x_1, x_4, x_{2w+3}\} \subset \mathbb{V}(C_n^q(1,2))$. We show that $\mathbb{R}$ is a resolving set for $C_n^q(1,2)$ (for $w = 2$ it is obvious, so we take $w \ge 3$). For this, we give the co-ordinates for every element of $\mathbb{V}(C_n^q(1,2))$ with respect to $\mathbb{R}$.

The co-ordinates for the vertices $\{x_j : j = 0, 1, 2, ..., n\}$ are

$$\gamma(x_{2k}|\mathbb{R}) = \begin{cases} (1, 1, w+1) & k = 1; \\ (k, k-2, w-k+2) & 2 \leq k \leq w+1; \\ (2w-k+2, w, 1) & k = w+2; \\ (2w-k+2, 2w-k+3, k-w-1) & w+3 \leq k \leq 2w+1 \end{cases}$$

and

$$\gamma(x_{2k+1}|\mathbb{R}) = \begin{cases} (0, 2, w) & k = 0; \\ (1, 1, w) & k = 1; \\ (k, k-1, w-k+1) & 2 \leq k \leq w; \\ (w, w, 0) & k = w+1; \\ (2w-k+1, 2w-k+3, k-w-1) & w+2 \leq k \leq 2w. \end{cases}$$

The co-ordinates for the vertices $\{y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq q\ \}$ are $\gamma(y_j^l|\mathbb{R}) = \gamma(x_j|\mathbb{R}) + (l, l, l)$ for $1 \leq j \leq n$ and $1 \leq l \leq q$.

From these codes, we find that $|\mathbb{X}| = |\mathbb{Y}^1| = |\mathbb{Y}^2| = ... = |\mathbb{Y}^l| = n$ and the sum of all of these cardinalities is equal to $|V(C_n^q(1,2))|$. Moreover, all of these sets are pairwise disjoint, and so we find that no pair of two distinct vertices in $C_n^q(1,2)$ are having the same metric codes, which implies that $dim(C_n^q(1,2)) \leq 3$, in this case. On the other hand, we show that $dim(C_n^q(1,2)) \geq 3$ by proving that there exists no resolving set $\mathbb{R}$ such that $|\mathbb{R}| = 2$. On the contrary, suppose $dim(C_n^q(1,2)) = 2$. By Theorem 2.3, we find that the valency of basis vertices can be 0, 1, 2, or 3. But except the vertices y_j^l $(1 \leq l \leq n$ and $1 \leq l \leq q)$, all other vertices of $C_n^q(1,2)$ have valency 5. Then, we have the following cases:

When the pair of vertices are in $\{y_j^l : 1 \leq l \leq n,\ 1 \leq l \leq q\}$ of the graph $C_n^q(1,2)$. Without loss of generality, we suppose that first resolving vertex is y_1^l $(1 \leq l \leq q)$. Suppose, second resolving vertex is y_j^l $(2 \leq j \leq 2w+2$ and $1 \leq l \leq q)$. Now, again we have two cases:

Subcase 2.1. $j \equiv 0\ mod(2)$: Then for $2 \leq j \leq 2w-2$ and $1 \leq l \leq q$, we have $\gamma(y_{4w+2}|\{y_1^l, y_j^l\}) = \gamma(x_{4w}|\{y_1^l, y_j^l\})$, when $j = 2w$ and $1 \leq l \leq q$, we have $\gamma(x_{4w}|\{y_1^l, y_j^l\}) = \gamma(x_{4w-1}|\{y_1^l, y_j^l\})$, and when $j = 2w+2$ and $1 \leq l \leq q$, we have $\gamma(x_2|\{y_1^l, y_j^l\}) = \gamma(x_{4w+2}|\{y_1^l, y_j^l\})$, a contradiction.

Subcase 2.2. $j \equiv 1\ mod(2)$: Then for $3 \leq j \leq 2w-1$ and $1 \leq l \leq q$, we have $\gamma(x_{4w+2}|\{y_1^l, y_j^l\}) = \gamma(x_{4w+1}|\{y_1^l, y_j^l\})$, and for $j = 2w+1$ and $1 \leq l \leq q$, we have $\gamma(x_2|\{y_1^l, y_j^l\}) = \gamma(x_{4w+1}|\{y_1^l, y_j^l\})$, a contradiction.

Hence, we find no resolving set with two vertices for $\mathbb{V}(C_n^q(1,2))$ implying that $dim(C_n^q(1,2)) = 3$, as well in this case.

Case 3. $n \equiv 3\ mod(4)$.

For this, we write $n = 4w+3$, $w \geq 2$, $w \in \mathbb{Z}^+$. Let $\mathbb{R} = \{x_1, x_2, x_{2w+2}\} \subset \mathbb{V}(C_n^q(1,2))$. We show that $\mathbb{R}$ is a resolving set for $C_n^q(1,2)$ (for $w = 2$ it is obvious, so we take $w \geq 3$). For this, we give the co-ordinates for every element of $\mathbb{V}(C_n^q(1,2))$ with respect to $\mathbb{R}$.

The co-ordinates for the vertices of $\{x_j : j = 0, 1, 2, ..., n\}$ are

$$\gamma(x_{2k}|\mathbb{R}) = \begin{cases} (k, k-1, w-k+1) & 1 \leq k \leq w+1; \\ (2w-k+2, 2w-k+3, k-w-1) & w+2 \leq k \leq 2w+1 \end{cases}$$

and

$$\gamma(x_{2k+1}|\mathbb{R}) = \begin{cases} (0, 1, w) & k = 0; \\ (k, k, w-k+1) & 1 \leq k \leq w; \\ (2w-k+2, 2w-k+2, k-w) & w+1 \leq k \leq 2w+1. \end{cases}$$

The co-ordinates for the vertices $\{y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq q\}$ are $\gamma(y_j^l|\mathbb{R}) = \gamma(x_j|\mathbb{R}) + (l, l, l)$ for $1 \leq j \leq n$ and $1 \leq l \leq q$.

From these codes, we find that $|\mathbb{X}| = |\mathbb{Y}^1| = |\mathbb{Y}^2| = ... = |\mathbb{Y}^l| = n$ and the sum of all of these cardinalities is equal to $|V(C_n^q(1,2))|$. Moreover, all of these sets are pairwise disjoint, and so we find that no pair of two distinct vertices in $C_n^q(1,2)$ are having the same metric codes, which implies that $dim(C_n^q(1,2)) \leq 3$, in this case. On the other hand, we show that $dim(C_n^q(1,2)) \geq 3$ by proving that there exists no resolving set $\mathbb{R}$ such that $|\mathbb{R}| = 2$. On the contrary, suppose $dim(C_n^q(1,2)) = 2$. By Theorem 2.3, we find that the valency of basis vertices can be 0, 1, 2, or 3. But except the vertices y_j^l $(1 \leq l \leq n$ and $1 \leq l \leq q)$, all other vertices of $C_n^q(1,2)$ have valency 5. Then, we have the following cases:

When the pair of vertices are in $\{y_j^l : 1 \leq l \leq n,\ 1 \leq l \leq q\}$ of the graph $C_n^q(1,2)$. Without loss of generality, we suppose that first resolving vertex is y_1^l $(1 \leq l \leq q)$. Suppose, second resolving vertex is y_j^l $(2 \leq j \leq 2w+3$ and $1 \leq l \leq q)$. Now, again we have two cases:

Subcase 3.1. $j \equiv 0\ mod(2)$: Then for $2 \leq j \leq 2w$ and $1 \leq l \leq q$, we have $\gamma(x_{4w+1}|\{y_1^l, y_j^l\}) = \gamma(y_{4w+3}|\{y_1^l, y_j^l\})$, and for $j = 2w+2$ and $1 \leq l \leq q$, we have $\gamma(x_2|\{y_1^l, y_j^l\}) = \gamma(x_{4w+2}|\{y_1^l, y_j^l\})$, a contradiction.

Subcase 3.2. $j \equiv 1\ mod(2)$: Then for $3 \leq j \leq 2w+1$ and $1 \leq l \leq q$, we have $\gamma(x_{4w+3}|\{y_1^l, y_j^l\}) = \gamma(x_{4w+2}|\{y_1^l, y_j^l\})$, and for $j = 2w+3$ and $1 \leq l \leq q$, we have $\gamma(x_3|\{y_1^l, y_j^l\}) = \gamma(x_{4w+2}|\{y_1^l, y_j^l\})$, a contradiction.

Hence, we find no resolving set with two vertices for $\mathbb{V}(C_n^q(1,2))$ implying that $dim(C_n^q(1,2)) = 3$, as well in this case.

Case 4. $n \equiv 1\ mod(4)$.

For this, we write $n = 4w+1$, $w \geq 2$, $w \in \mathbb{Z}^+$. Let $\mathbb{R} = \{x_1, x_2, x_3, x_{2w+2}\} \subset \mathbb{V}(C_n^q(1,2))$. We show that $\mathbb{R}$ is a resolving set for $C_n^q(1,2)$ (for $w = 2$ it is obvious, so we take $w \geq 3$). For this, we give the co-ordinates for every element of $\mathbb{V}(C_n^q(1,2))$ with respect to $\mathbb{R}$.

The co-ordinates for the vertices $\{x_j : j = 1, 2, ..., n\}$ are

$$\gamma(x_{2k}|\mathbb{R}) = \begin{cases} (1, 0, 1, w) & k = 1; \\ (k, k-1, k-1, w-k+1) & 2 \leq k \leq w; \\ (w, k-1, k-1, 0) & k = w+1; \\ (2w-k+1, 2w-k+1, 2w-k+2, k-w-1) & w+2 \leq k \leq 2w \end{cases}$$

and

$$\gamma(x_{2k+1}|\mathbb{R}) = \begin{cases} (0,1,1,w) & k=0; \\ (k,k,k-1,w-k+1) & 1 \le k \le w; \\ (w,w,w,1) & k=w+1; \\ (2w-k+1,2w-k+1,2w-k+2,k-w) & w+2 \le k \le 2w. \end{cases}$$

The co-ordinates for the vertices $\{y_j^l : 1 \le j \le n,\ 1 \le l \le q\ \}$ are $\gamma(y_j^l|\mathbb{R}) = \gamma(x_j|\mathbb{R}) + (l,l,l,l)$ for $1 \le j \le n$ and $1 \le l \le q$.

Again from these codes, we find that $|\mathbb{X}| = |\mathbb{Y}^1| = |\mathbb{Y}^2| = ... = |\mathbb{Y}^l| = n$ and the sum of all of these cardinalities is equal to $|V(C_n^q(1,2))|$. Moreover, all of these sets are pairwise disjoint, and so we find that no pair of two distinct vertices in $C_n^q(1,2)$ are having the same metric codes, which implies that $dim(C_n^q(1,2)) \le 4$, in this case. Conversely, to complete the proof, we show that $dim(C_n^q(1,2)) \ge 4$. In [2], Borchert and Gosselin proved that $dim(C_n(1,2)) = 4$ if $n \equiv 1(mod4)$ and $dim(C_n(1,2)) = 3$ otherwise. Buczkowski et al. [3], proved that if H is a graph obtained from a nontrivial connected graph G by adding a pendant edge to G, then $dim(G) \le dim(H) \le dim(G)+1$. From this, we find that $dim(C_n^q(1,2)) \ge 4$ for $q=1$ and so repeating this q times we always have $dim(C_n^q(1,2)) \ge 4$ for every $1 \le l \le q$, which concludes the proof in this case. ■

For $q=1$, we call the graph $C_n^q(1,2)$ as the circulant graph with pendant edges (see Fig. 2). Then, by Theorem 3.1, we have the following corollary:

Corollary 3.2. *For $n \ge 8$, we have*

$$dim(C_n^1(1,2)) = \begin{cases} 3 & if\ \ n \equiv 0,2,3\ (mod4); \\ 4 & if\ \ n \equiv 1\ (mod4). \end{cases}$$

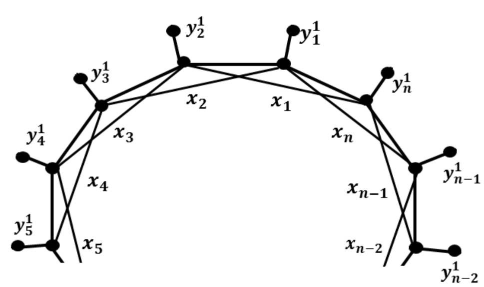

Figure 2: The graph $C_n^1(1,2)$

4. Conclusion

In this article, we have studied the metric dimension of the graph $C_n^q(1, 2)$, which is obtained from the circulant graph $C_n(1, 2)$ by joining n-path of length q at each vertex of the graph $C_n(1, 2)$. We proved that, $dim(C_n^q(1, 2)) = 3$, for $n \equiv 0, 2, 3 \ mod(4)$ and $dim(C_n^q(1, 2)) = 4$, for $n \equiv 1 \ mod(4)$. We also observed that $dim(C_n(1, 2)) = dim(C_n^q(1, 2))$, for every $n \geq 8$ and $q \geq 1$.

References

[1] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffman, M. Mihalak, L.S. Ram, Network discovery and verification, *IEEE Journal on Selected Areas in Communications* **24** (2006) 2168–2181.

[2] A. Borchert, S. Gosselin, The metric dimension of circulant graphs and Cayley hypergraphs, *Utilitas Mathematica* **106** (2018) 125–147.

[3] P.S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, *Periodica Mathematica Hungarica* **46** (1) (2003) 9–15.

[4] G. Chartrand, J. Eroh, M.A. Johnson, O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Applied Mathematics* **105** (2000) 99–113.

[5] F. Harary, R.A. Melter, On the metric dimension of a graph, *Ars Combinatoria* **2** (1976) 191–195.

[6] M. Imran, A.Q. Baig, S.A. Bokhary, I. Javaid, On the metric dimension of circulant graphs, *Applied Mathematics Letters* **25** (2012) 320–325.

[7] M. Imran, S.A. Bokhary, A.Q. Baig, Families of rotationally-symmetric plane graphs with constant metric dimension, *Southeast Asian Bull. Math.* **36** (2012) 663–675.

[8] I. Javaid, M.T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Mathematica* **75** (2008) 21–33.

[9] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Applied Mathematics* **70** (1996) 217–229.

[10] R.A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing* **25** (1984) 113–121.

[11] F.P. Muga II, On hamiltonian decomposition embedding and diameter of certain circulant graphs, *Southeast Asian Bull. Math.* **23** (1) (1999) 117–126.

[12] S.K. Sharma, V.K. Bhat, Metric dimension of heptagonal circular ladder, *Discrete Mathematics, Algorithms and Applications* **13** (1) (2021) (2050095), 17 pages.

[13] S.K. Sharma, V.K. Bhat, Fault-tolerant metric dimension of two-fold heptagonal-nonagonal circular ladder, *Discrete Mathematics, Algorithms and Applications* **14** (3) (2021), 23 pages.

[14] S.K. Sharma, V.K. Bhat, On some plane graphs and their metric dimension, *International Journal of Applied and Computational Mathematics* **7** (2021), 203, 25 pages.

[15] P.J. Slater, Leaves of trees, *Congressus Numerantium* **14** (1975) 549–559.

[16] I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bulletin mathmatique de la Socit des Sciences Mathmatiques de Roumanie* **50** (2007) 371–376.

[17] T. Vetrk, The metric dimension of circulant graphs, *Canadian Mathematical Bulletin* **60** (1) (2017) 206–216.

[18] L. Xiao, Y. Liu, Even regular factor of regular graphs and number of cut edges, *Southeast Asian Bull. Math.* **31** (5) (2007) 1019–1026.