Geometry in A-Metric Space

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Abstract. In this paper, we study the geometry of an A-metric space of Subba Rao [15], by introducing the concepts of metric betweenness and its properties t_1, t_2, B-linearity and D-linearity. It is proved that there do not exist equilateral triangles in any A-metric space (A, A, d), where A is any representable autometrized algebra satisfying (R) and (S). It is also proved that any A-metric space is Ptolemaic.

1. Introduction

In this paper, we study the geometry of an A-metric space of Subba Rao [15], where A satisfies the conditions:

(R): $a * (a \wedge b) + (a \wedge b) = a$.

(S): $(a * (a \vee b)) \wedge (b * (a \vee b)) = 0$, for all a, b in A.

Blumenthal [2] and Penning [6] studied the metric betweenness in a Boolean metric space and Swamy [17] studied the metric betweenness in l-groups. Subba

Rao studied the geometry of a representable autometrized algebra satisfying R and S. Thus, Subba Rao generalized the geometric results of Swamy [17, 19] for commutative l-groups and those of Blumenthal [2] and Ellis [4] for Boolean algebras.

Ranga Rao [8] briefly discussed the geometry of a G-metric space (X, G, d). By introducing the concept of the metric betweenness in a G-metric space, He obtained several results regarding the properties of metric betweenness, B-linearity, D-linearity in a G-metric space. He proved that any G-metric space is Ptolemaic. Following Ranga Rao [8], we introduce the notion of metric betweenness, B-linearity, D-linearity in an A-metric space. And, we obtain some interesting geometric properties of an A-metric space, thus generalizing the geometric results of Ranga Rao [8]. We also introduce the concept of a triangle in an A-metric space and obtained the interesting result that there do not exist equilateral triangles in any A-metric space, where A is a representable autometrized algebra satisfying (R) and (S).

2. A-metric Space

Definition 2.1. *A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a representable autometrized algebra, if and only if, A satisfies the following conditions:*

(i) *$A = (A, +, \leq, *)$ is a semi regular autometrized algebra;*
(ii) *for every $a \in A$, all the mappings $x \longmapsto a + x$, $x \longmapsto a \vee x$, $x \longmapsto a \wedge x$ and $x \longmapsto a * x$ are contractions (A mapping $f : A \to A$ is a contraction w.r.to $*$, if and only if, $f(x) * f(y) \leq x * y$ for all x, y in A).*

Hereafter $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the following identities:

(R): $a * (a \wedge b) + (a \wedge b) = a$.
(S): $(a * (a \vee b)) \wedge (b * (a \vee b)) = 0$.

Definition 2.2. [14, 15] *Let X be a non empty set, let $A = (A, +, *, \leq)$ be a representable autometrized algebra, let $d : X \times X \longrightarrow A$ be a mapping satisfying the following conditions of a distance function:*

(M_1): *$d(a, b) \geq 0$ for all a, b in X, with equality occurring, if and only if, $a = b$ (non-negativity).*
(M_2): *$d(a, b) = d(b, a)$ for all a, b in X (symmetry).*
(M_3): *$d(a, c) \leq d(a, b) + d(b, c)$ for all a, b, c in X (triangle inequality).*

Then, (X, A, d) is said to be an A-metric space.

Now, we introduce the "metric betweenness" in an A-metric space.

Definition 2.3. *Let (X, A, d) be an A-metric space. An element x of X is said to lie "metrically between a and b of X", if and only if, $d(a, x) + d(x, b) = d(a, b)$, and in this case we write $B(a, x, b)$.*

Theorem 2.4. *In any A-metric space(X, A, d), we have $B(a, b, c)$, if and only if, $B(c, b, a)$ (symmetry in the outer points), for any $a, b, c \in X$.*

Proof. Let (X, A, d) be an A-metric space, and $a, b, c \in X$.

Now, we have

$$
\begin{aligned}
B(a, b, c) &\Leftrightarrow d(a, b) + d(b, c) = d(a, c) \\
&\Leftrightarrow d(b, c) + d(a, b) = d(a, c) \\
&\Leftrightarrow d(c, b) + d(b, a) = d(c, a), \text{(by the symmetry of } d) \\
&\Leftrightarrow B(c, b, a)
\end{aligned}
$$
∎

Definition 2.5. *Any betweenness relation B is said to satisfy the special inner point property $(\beta):$ $B(a, b, c)$ and $B(a, c, b)$, if and only if, $b = c$.*

Following the terminology introduced in Nordhaus and Lapidus [5], we shall treat any three elements a, b, c of an A-metric space (X, A, d) as the vertices of a triangle with sides of length $d(a, b), d(b, c)$ and $d(c, a)$ and we denote this triangle by $\Delta(a, b, c)$. Further, if any two sides of a triangle $\Delta(a, b, c)$ are equal, then it is said to be an Isosceles triangle. If all the three sides are equal, then it is said to be equilateral triangle in the space (X, A, d).

Subba Rao [11] proved that, in any representable autometrized algebra $(A, +, \leq, *)$ satisfying (R) and (S), three points a, b, c fail to have the special inner point property, if a, b, c are all the vertices of an Isosceles triangle in which the sum of any two sides is equal to their union (join). We extend this result of Subba Rao to an A-metric space in the following theorem.

Theorem 2.6. *Let (X, A, d) be an A-metric space. Three points a, b, c in X fail to have the special inner point property, if a, b, c are the vertices of an Isosceles triangle, in which the sum of any two sides is equal to their union (join).*

Proof. Let (X, A, d) be an A-metric space, and a, b, c be any three elements in A. Assume that these are the vertices of an Isosceles triangle, in which the sum of any two sides is equal to their join, specifically, let us assume that

$$
\begin{aligned}
d(a, b) &= d(a, c), & (1) \\
d(a, b) + d(b, c) &= d(a, b) \vee d(b, c) & (2) \\
d(a, b) + d(a, c) &= d(a, b) \vee d(a, c) & (3) \\
d(a, c) + d(b, c) &= d(a, c) \vee d(b, c) & (4)
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
d\,(a,b) + d\,(b,c) &= d\,(a,b) \vee d\,(b,c) \\
&\leq d\,(a,b) \vee [d\,(b,a) + d\,(a,c)] \quad (\text{by} \quad (M_3)) \\
&= d\,(a,b) \vee [d\,(a,b) + d\,(a,c)] \\
&= d\,(a,b) \vee [d\,(a,b) \vee d\,(a,c)] \quad (\text{by} \ (3)\ ) \\
&= d\,(a,b) \vee d\,(a,c) \quad (\text{by idempotency and associativity of } \vee) \\
&= d\,(a,c) \vee d\,(a,c) \quad (\text{by} \ (1)) \\
&= d\,(a,c).
\end{aligned}
$$

Thus,

$$d\,(a,b) + d\,(b,c) \leq d\,(a,c). \tag{5}$$

But,

$$d\,(a,b) + d\,(b,c) \geq d\,(a,c) \quad (\text{triangle inequality}) \tag{6}$$

From (5) and (6), it follows that

$$d\,(a,b) + d\,(b,c) = d\,(a,c) \tag{7}$$

Thus, we have

$$B\,(a,b,c) \quad \text{holds good} \tag{8}$$

Now, we have

$$
\begin{aligned}
d\,(a,c) + d\,(c,b) &= d\,(a,b) + d\,(c,b) \\
&= d\,(a,b) + d\,(b,c) \\
&= d\,(a,c) \quad \text{from (iii)} \\
&= d\,(a,b) \quad (\text{from I})
\end{aligned}
$$

Thus,

$$B\,(a,c,b) \quad \text{holds} \tag{9}$$

From (5) and (6), we have $B\,(a,b,c)$ and $B\,(a,c,b)$ where $b \neq c$, i.e., the special inner point property fails in this case. Hence the proof is completed. ∎

Following Subba Rao [11], we introduce below the transistivity t_1 and t_2 for a betweeness relation in any A-metric space.

Definition 2.7. *Any betweenness relation B is said to have*
 (i) *the transitivity t_1, if and only if, $B\,(a,b,c)$, $B\,(a,d,b) \Rightarrow B\,(d,b,c)$.*

(ii) *the transitivity t_2, if and only if, $B(a, b, c)$, $B(a, d, b) \Rightarrow B(a, d, c)$.*

Theorem 2.8. *If the metric betweenness B has transitivity t_1 in an A-metric space(X, A, d), then X is free of Isosceles triangles, in which, the sum of any two sides is equal to their union (join).*

Proof. Let (X, A, d) be an A-metric space, let B denote the metric betweenness in the space X. Assume that B has transitivity t_1, i.e., $B(a, b, c)$, $B(a, d, b)$ $\Rightarrow B(a, b, c)$.

If possible, suppose that there exist an Isosceles triangle $\Delta(a, b, c)$ in X, where $d(a, b) = d(a, c)$ and the sum of any two sides is equal to their join.

Then B fails to have the special inner point property for the points a, b, c in X. And we have $B(a, b, c)$ and $B(a, c, b)$ holds good, where $b \neq c$.

By the transtivity t_1 of B, it follows that

$$B(c, b, c) \qquad \text{hold.}$$
$$\Rightarrow d(c, b) + d(b, c) = d(c, c)$$
$$\Rightarrow d(b, c) + d(b, c) = 0 \qquad \text{by } (M_1 \text{ and } M_2)$$
$$\Rightarrow 2d(b, c) = 0 \qquad \text{where } b \neq c$$
$$\Rightarrow d(b, c) \leq d(b, c) + 0 \leq d(b, c) + d(b, c) = 2d(b, c) = 0$$
$$\Rightarrow d(b, c) \leq 0$$
$$\Rightarrow b = c \qquad \text{(since } d(b, c) \geq 0 \text{ by } (M_1)),$$

which is a contradiction to the fact that $b \neq c$ (since a, b, c are the vertices of a Isosceles triangle). Thus, our supposition is false. Hence, X is free of Isosceles triangles for which the sum of any two sides of a triangle is equal to their join. ∎

Theorem 2.9. *In any A-metric space (X, A, d), the metric betweenness B has the transitivity t_2.*

Proof. Let a, b, c and d be any elements in X. We have to prove that

$$B(a, b, c), B(a, d, b) \Rightarrow B(a, d, c)$$

Assume that $B(a, b, c)$ and $B(a, d, b)$. Now,

$$B(a, b, c)$$
$$\Rightarrow d(a, b) + d(b, c) = d(a, c) \qquad (10)$$
$$B(a, d, b)$$
$$\Rightarrow d(a, d) + d(d, b) = d(a, b) \qquad (11)$$

From (11), we get

$$d(a, d) + d(d, b) + d(b, c) = d(a, b) + d(b, c)$$
$$\Rightarrow d(a, d) + d(d, b) + d(b, c) = d(a, c) \qquad \text{(from (10))}$$
$$\Rightarrow d(a, d) + d(d, c) \leq d(a, c) \qquad (12)$$

But

$$d(a, c) \leq d(a, d) + d(d, c) \qquad \text{(from triangle inequality)} \qquad (13)$$

From (12) and (13), it follows that

$$d(a, d) + d(d, c) = d(a, c)$$
$$\Rightarrow B(a, d, c)$$

Thus, $B(a, b, c), B(a, d, b) \Rightarrow B(a, d, c)$. Therefore B has the transitivity t_2. $\blacksquare$

Theorem 2.10. *In the A-metric space (A, A, d), lattice betweeness implies metric betweenness, i.e., $L(a, b, c) \Rightarrow B(a, b, c)$, for any a, b, c in A.*

Proof. Let $A = (A, +, \leq, *)$ be a representable autometrized algebra satisfying (R) and (S). Therefore A satisfies the following properties:

(i) $a * b = (a \vee b) * (a \wedge b)$, for all a, b in A.
(ii) $a * b + b * c = (a \wedge c) * b + b * (a \vee c)$, for all a, b, c in A.
(iii) $a \leq b \leq c \Rightarrow B(a, b, c)$.

Now, consider the A-metric space(A, A, d), where $d(a, b) = a * b$ for all a, b in A.

Let a, b, c belong to A. Assume that $L(a, b, c)$. Then we have

$$a \wedge c \leq b \leq a \vee c$$
$$\Rightarrow B(a \wedge c, b, a \vee c) \qquad \text{from (iii)}$$
$$\Rightarrow (a \wedge c) * b + b * (a \vee c) = (a \wedge c) * (a \vee c)$$
$$\Rightarrow a * b + b * c = a * c, \qquad \text{from (i) and (ii)}$$
$$\Rightarrow B(a, b, c)$$

Thus, $L(a, b, c) \Rightarrow B(a, b, c)$. $\blacksquare$

Definition 2.11. *An n-tuple of distinct points of an A-metric space (X, A, d), $(n \geq 3)$ is said to be*

(i) *B-Linear, if and only if, there exists a labeling $(p_1, p_2, p_3, ..., p_n)$ of its elements such that $B(p_i, p_j, p_k)$ holds whenever $1 \leq i < j < k \leq n$.*
(ii) *D-Linear, if and only if, there exists a labelling $(p_1, p_2, p_3, ..., p_n)$ of its elements such that$d(p_1, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})$.*

Theorem 2.12. *B-linearity implies D-linearity in the A-metric space (X, A, d).*

Proof. Consider the A-metric space (X, A, d), where A is a representable autometrized algebra satisfying (R) and (S). Assume that an n-tuple P of n elements of A is B-linear, $(n \geq 3)$.

Therefore, there exists a labeling $(p_1, p_2, p_3, ..., p_n)$ of elements of P such that $B(p_i, p_j, p_k)$ holds whenever $1 \leq i < j < k \leq n$; In particular, we have, for any $i < n - 1$, $B(p_i, p_{i+1}, p_n)$ holds. Then we have

$$d(p_i, p_{i+1}) + d(p_{i+1}, p_n) = d(p_i, p_n)$$
$$\Rightarrow d(p_i, p_n) = d(p_i, p_{i+1}) + d(p_{i+1}, p_n)$$

Hence, we have

$$
\begin{aligned}
d(p_1, p_n) &= d(p_1, p_2) + d(p_2, p_n), \qquad (1 < n < 2) \\
&= d(p_1, p_2) + d(p_2, p_3) + d(p_3, p_n) \\
&= d(p_1, p_2) + d(p_2, p_3) + d(p_3, p_4) + d(p_4, p_n) \\
&\;\;\vdots \\
&= \sum_{i=1}^{n-1} d(p_i, p_{i+1})
\end{aligned}
$$

Thus,

$$d(p_1, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})$$
$$\Rightarrow P \text{ is } D\text{-linear.}$$

The proof is completed.   ∎

Corollary 2.13. *B-Linearity implies D-Linearity in the A-metric space* (A, A, d).

Proof. The proof is obvious.   ∎

Remark 2.14. The above corollary is true in any A-metric space (X, A, d), where X is either a Boolean algebra or a commutative l-group, in view of the fact that lattice betweenness and metric betweenness are equivalent in Boolean algebras and commutative l-groups. Further, it may be noted that the above corollary is true in the case of any A-metric space, where X itself is a representable algebra satisfying (R) and (S), not necessarily identical with A.

Blumenthal [2] proved that any Boolean geometry is Ptolemaic. Swamy [17] had shown that any lattice ordered autometrized algebra is Ptolemaic; Ranga Rao [8] proved that any G-metric space is Ptolemaic.

Definition 2.15. *An A-metric space* (X, A, d) *is said to be Ptolemaic, if and only if, for every four points* a, b, c, d *in* X, *it is true that*

$$
\begin{aligned}
d(a, b) \wedge d(c, d) &\leq d(a, c) \wedge d(b, d) + d(a, d) \wedge d(b, c), \\
d(a, c) \wedge d(b, d) &\leq d(a, b) \wedge d(c, d) + d(a, d) \wedge d(c, b), \\
d(a, d) \wedge d(b, c) &\leq d(a, c) \wedge d(d, b) + d(a, b) \wedge d(c, d).
\end{aligned}
$$

Theorem 2.16. *Any A-metric space* (X, A, d) *is Ptolemaic.*

Proof. Let (X, A, d) be any A-metric space, let $a \in X$, $b \in X$, $c \in X$ and $d \in X$ be any four points. Then we have

$$d(a, c) \wedge d(b, d) + d(a, d) \wedge d(b, c)$$
$$= [d(a, c) + [d(a, d) \wedge d(b, c)]] \wedge [d(b, d) + [d(a, d) \wedge d(b, c)]]$$
$$= [d(a, c) + d(a, d)] \wedge [d(a, c) + d(b, c)] \wedge [d(b, d) + d(a, d)] \wedge [d(b, d) + d(b, c)]$$
$$\geq d(c, d) \wedge d(a, b) \wedge d(b, a) \wedge d(d, c)$$
$$\geq d(c, d) \wedge d(a, b)$$
$$= d(a, b) \wedge d(c, d).$$

Therefore, $d(a, b) \wedge d(c, d) \leq d(a, c) \wedge d(b, d) + d(a, d) \wedge d(b, c)$.

Similarly the other two inequalities can be proved. Therefore (X, A, d) is Ptolemaic. ∎

In any Boolean geometry, since the metric operation (the symmetric difference) is a group operation, there do not exist any Isosceles triangles.

In commutative l-groups also Swamy [17] had shown that equilateral triangles do not exist. Subba Rao [12] observed that there do not exist equilateral triangles in any representable autometrized algebras satisfying (R) and (S).

Theorem 2.17. *There do not exist equilateral triangles in the A-metric space* (A, A, d), *where A is any representable autometrized algebra satisfying* (R) *and* (S).

Proof. Let (A, A, d) be an A-metric space, where A is a representable autometrized algebra satisfying (R) and (S). Here $d(x, y) = x * y$, since there do not exist equilateral triangles in A (see [12]), we cannot find pair wise distinct elements a, b, c in A such that $d(a, b) = d(b, c) = d(c, a)$. Thus, there do not exist equilateral triangles in the A-metric space (A, A, d). ∎

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