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Multipliers in Almost Semilattices

Terefe Getachew Beyene and Habtamu Tiruneh Alemu
Department of Mathematics, Addis Ababa Science and Technology University,
Addis Ababa, Ethiopia
Email: terefe.getachew@aastu.edu.et; gterefe14@gmail.com;
habtamu.tiruneh@aastu.edu.et; htiruneh4@gmail.com

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Abstract. The concept of multipliers in an Almost Semilattice(ASL) is introduced and some basic properties are proved. More over, in an ASL L, we introduced a congruence relation ϕ_a for $a \in L$ and some useful properties of ϕ_a are derived.

Keywords: Semilattices; Almost semilattice; Almost semilattice with zero; Multiplier; Isoton; Idempotent; $Fix_f(L)$.

1. Introduction

The notion of derivation, introduced from analytic theory, is helpful to the research of structure and property in algebraic system. Recently, analytic and algebraic properties of lattice were widely researched [3, 5]. Several authors [1, 4] have studied derivations in rings and near-rings after Posner [9] have given the definition of the derivation in ring theory. Bresar [2] introduced the generalized derivation in rings and many mathematicians studied on this concept. K.L. Xin et al. applied the notion of the derivation in ring theory to lattices [10]. In [7], a partial multiplier on a commutative semigroup $(A, .)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x).y = x.F(y)$ for all $x, y \in D_F$. Moreover, in [6] K.H. Kim introduced the concept of multipliers in an almost distributive lattice. The concept of Almost Semilattice (ASL) was introduced by G. Nanaji Rao and T.G. Beyene [8] as a generalization of almost distributive lattice and a semilattice.

In this paper, we introduced the notion of multipliers in Almost Semilattice(ASL) and proved some related properties. Moreover, in an ASL L, we introduced a congruence relation ϕ_a for $a \in L$ and derived some useful properties of ϕ_a.

2. Preliminaries

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. *Let $(P, \leq)$ be a poset and $a \in P$. Then*

(1) *a is called the least element of P if $a \leq x$ for all $x \in P$.*
(2) *a is called the greatest element of P if $x \leq a$ for all $x \in P$.*

It can be easily observed that, if the least (greatest) element exists in a poset, then it is unique.

Definition 2.2. *Let $(P, \leq)$ be a poset and $a \in P$. Then*

(1) *a is called a minimal element, if $x \in P$ and $x \leq a$, then $x = a$.*
(2) *a is called maximal element, if $x \in P$ and $a \leq x$, then $a = x$.*

It can be easily verified that the least (greatest) element (if exists), then it is minimal (maximal), but the converse need not be true.

Definition 2.3. *Let $(P, \leq)$ be a poset and S be a non empty subset of P. Then*

(1) *An element a in P is called a lower bound of S if $a \leq x$ for all $x \in S$.*
(2) *An element a in P is called an upper bound of S if $x \leq a$ for all $x \in S$.*
(3) *An element a in P is called the greatest lower bound (g.l.b or infimum) of S if a is a lower bound of S and $b \in P$ such that b is a lower bound of S, then $b \leq a$.*
(4) *An element a in P ia called the least upper bound (l.u.b or supremum) of S if a is an upper bound of S and $b \in P$ such that b is an upper bound of S, then $a \leq b$.*

Definition 2.4. *A Semilattice L is an algebra $(L, \star)$ of type (2) satisfies the following conditions:*

(1) $(x \star y) \star z = x \star (y \star z)$ *(Associative Law)*
(2) $x \star y = y \star x$ *(Commutative Law)*
(3) $x \star x = x$, *for all $x, y, z \in L$.* *(Idempotent)*

Definition 2.5. *An Almost Semilattice(ASL) is an algebra $(L, \circ)$ of type (2) satisfies the following conditions:*

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ *(Associative Law)*
(2) $(x \circ y) \circ z = (y \circ x) \circ z$ *(Almost Commutative Law)*
(3) $x \circ x = x$, *for all* $x, y, z \in L$. *(Idempotent)*

Definition 2.6. *Let L be an ASL. Then for any $a, b \in L$, we say that a is less or equal to b and write $a \leq b$, if $a \circ b = a$.*

Lemma 2.7. *Let L be an ASL. Then for any $a, b \in L$, we have:*

(1) $a \circ (a \circ b) = a \circ b$
(2) $(a \circ b) \circ b = a \circ b$
(3) $b \circ (a \circ b) = a \circ b$.

Corollary 2.8. *Let L be an ASL. Then for any $a, b \in L, a \circ b \leq b$.*

Corollary 2.9. *Let L be an ASL. Then for any $a, b \in L, a \circ b = b \circ a$ whenever $a \leq b$.*

Theorem 2.10. *Let L be an ASL. Then the relation $\leq$ is a partial ordering on L.*

Definition 2.11. *Let L be an ASL. An element $a \in L$ is said to be minimal (maximal) element if for any $x \in L$, $x \leq a (a \leq x)$, then $x = a (a = x)$.*

Theorem 2.12. *Let L be an ASL. Then for any $a, b \in L$ with $a \leq b$. Then $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$ for all $c \in L$.*

Theorem 2.13. *Let L be an ASL. Then for any $a, b, c \in L$, we have the following statements:*

(1) $a \leq b$ *and* $c \leq d \implies a \circ c \leq b \circ d$.
(2) $a, b \leq c \implies a \circ b, b \circ a \leq c$
(3) $a \leq b, c \implies a \leq b \circ c, c \circ b$.

Definition 2.14. *Let L be an ASL. An element $0 \in L$ is called a zero element of L if $0 \circ a = 0$ for all $a \in L$.*

It can be easily seen that an ASL can have at most one zero element and it will be the least element of the poset $(L, \leq)$. We always denote the zero element of L, if it exists, by '0'. If L has an element 0 and satisfies the property $0 \circ x = 0$ for all $x \in L$ along with Definition 2.5, then L is called an ASL with '0'.

Lemma 2.15. *Let L be an ASL with 0. Then for any $a \in L, a \circ 0 = 0$.*

Lemma 2.16. *Let L be an ASL with 0. Then for any $a, b \in L, a \circ b = 0$ if and only if $b \circ a = 0$.*

Corollary 2.17. *Let L be an ASL with 0. Then for any $a, b \in L, a \circ b = b \circ a$ whenever $a \circ b = 0$.*

3. Multipliers in Almost Semilattices

In this section, we introduce the concept of multipliers in Almost Semilattice(ASL) and give some examples of it. We also, prove some related properties of multipliers in Almost semilattices. Moreover, in Almost semilattices, we establish a congruence relation ϕ_a for $a \in L$ and derive some useful properties of ϕ_a in L. In what follows, L denotes an Almost semilattice, unless otherwise specified.

Definition 3.1. *Let L be an ASL. A function $f : L \longrightarrow L$ is called a multiplier of L if $f(x \circ y) = f(x) \circ y$ for all $x, y \in L$.*

Example 3.2. Let $L = \{a, b, c\}$. Define a binary operation $\circ$ on L as follows:

$\circ$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

Then clearly $(L, \circ)$ is an ASL. Now, define $f : L \longrightarrow L$ by

$$f(x) = \begin{cases} a & \text{if } x = a, \\ b & \text{if } x = b,\ c. \end{cases}$$

Then f is a multiplier on L.

Lemma 3.3. *Let L be an ASL. Then the identity map on L is a multiplier on L. This multiplier is called identity multiplier.*

Proof. Let f be the identity map on L. Then for any $x, y \in L$, $f(x \circ y) = x \circ y = f(x) \circ y$. Therefore f is a multiplier on L. ∎

Theorem 3.4. *Every multiplier in a discrete ASL is an identity multiplier.*

Proof. Let L be a discrete ASL and f be a multiplier of L. Then for any $a \in L$, $f(a) = f(a \circ a) = f(a) \circ a = a$ (since L is discrete ASL). Therefore f is an identity multiplier. ∎

Example 3.5. Let L be a ASL with 0. Then a function $f : L \longrightarrow L$ defined by $f(x) = 0$ for any $x \in L$ is a multiplier on L which is called a zero multiplier.

Example 3.6. Let $L = \{0, a, b, c\}$. Define a binary operation $\circ$ on L as follows:

∘	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(L, \circ, 0)$ is an ASL with 0. Now, define $f : L \longrightarrow L$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0,\ c, \\ a & \text{if } x = a,\ b. \end{cases}$$

Then f is a multiplier on L.

Remark 3.7. In an almost semilattice, every function may not be a multiplier.

Example 3.8. In a discrete ASL $L = \{0, a, b\}$, if we define $f : L \longrightarrow L$ by $f(0) = 0$, $f(a) = b$ and $f(b) = a$. Then f is not a multiplier on L since $f(a \circ b) = f(b) = a \neq b = f(a) \circ b$.

Lemma 3.9. *Let f be a multiplier on an ASL L. Then the following conditions hold:*

(1) *$f(x) \leq x$ for all $x \in L$.*
(2) *$f(x) \circ f(y) \leq f(x \circ y)$ for all $x,\ y \in L$.*
(3) *If L has 0, then $f(0) = 0$.*
(4) *$f^2(x) = f(f(x)) = f(x)$ for all $x \in L$.*
(5) *If I is an ideal of L, then $f(I) \subseteq I$.*

Proof. (1) Let $x \in L$. Then $f(x) = f(x \circ x) = f(x) \circ x$. Therefore $f(x) \leq x$ for all $x \in L$.

(2) Let $x, y \in L$. Then $f(x \circ y) = f(x) \circ y$. Now, by (1) $f(x) \circ f(y) \leq f(x) \circ y = f(x \circ y)$. Therefore $f(x) \circ f(y) \leq f(x \circ y)$ for all $x,\ y \in L$.

(3) By (1), $f(0) \leq 0$. Thus $0 \leq f(0) \leq 0$ and hence $f(0) = 0$.

(4) For any $x \in L$, by (1), we have $f(x) = f(x) \circ x = x \circ f(x)$. Hence $f^2(x) = f(f(x)) = f(x \circ f(x)) = f(x) \circ f(x) = f(x)$, (since $f : L \to L$ is a multiplier of L and using (1) again).

(5) Let I be an ideal of L and $a \in I$. Then by (1) again, $f(a) \leq a$ and since I is an initial segment $f(a) \in I$. Therefore $f(I) \subseteq I$. ∎

Lemma 3.10. *Let L be an ASL and $a \in L$. Define a function $f_a : L \longrightarrow L$ by $f_a(x) = a \circ x$ for all $x \in L$. Then f_a is a multiplier on L. Such a multiplier of L is called a principal multiplier of L.*

Proof. Let $x, y \in L$. Then $f_a(x \circ y) = a \circ (x \circ y) = (a \circ x) \circ y = f_a(x) \circ y$ for all $x, y \in L$. Therefore f_a a multiplier on L. ■

Definition 3.11. *A multiplier f in an ASL L is called an isotone multiplier if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in L$.*

Lemma 3.12. *Let L be an ASL and f be a multiplier of L. If $x \leq y$ and $f(y) = y$, then $f(x) = x$.*

Proof. Let $x \leq y$ and $f(y) = y$. Then $f(x) = f(x \circ y) = f(y \circ x) = f(y) \circ x = y \circ x = x \circ y = x$. ■

Theorem 3.13. *Let L be an ASL and f be a multiplier of L. Then f is an isotone multiplier.*

Proof. Let $x, y \in L$ such that $x \leq y$. Then $f(x) = f(x \circ y) = f(y \circ x) = f(y) \circ x \leq f(y) \circ y = f(y)$ (since by Lemma 3.9). Hence $f(x) \leq f(y)$. Therefore f is an isotone multiplier. ■

Corollary 3.14. *Let L be an ASL. Then every principal multiplier on L is an isotone multiplier on L.*

Proof. Let $x, y \in L$ such that $x \leq y$. Then $f_a(x) = f_a(x \circ y) = a \circ x \circ y = a \circ x \circ a \circ y = f_a(x) \circ f_a(y)$. Hence $f_a(x) \leq f_a(y)$. Therefore f_a is an isotone multiplier of L. ■

Proposition 3.15. *Let L be an ASL and f_1 and f_2 be two multipliers of L. Then the composition map of f_1 and f_2, $f_1 \circ f_2$ is a multiplier of L.*

Proof. Let $x, y \in L$. Then $(f_1 \circ f_2)(x \circ y) = f_1(f_2(x \circ y)) = f_1(f_2(x) \circ y) = f_1(f_2(x)) \circ y = (f_1 \circ f_2)(x) \circ y$. Therefore $f_1 \circ f_2$ is a multiplier of L. ■

Proposition 3.16. *Let L_1 and L_2 be two ASLs with 0. Define $f : L_1 \times L_2 \longrightarrow L_1 \times L_2$ by $f(x, y) = (0, y)$ for all $(x, y) \in L_1 \times L_2$. Then f is a multiplier of $L_1 \times L_2$ with respect to pointwise operation.*

Proof. Let L_1 and L_2 be two ASLs with 0. Then clearly, $(L_1 \times L_2, \circ, (0, 0))$ is an ASL with zero under the pointwise operation, where $(x_1, y_1) \circ (x_2, y_2) = (x_1 \circ y_1, x_2 \circ y_2)$ for any $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$. Now, $f((x_1, y_1) \circ (x_2, y_2)) = f(x_1 \circ x_2, y_1 \circ y_2) = (0, y_1 \circ y_2) = (0 \circ x_2, y_1 \circ y_2) = (0, y_1) \circ (x_2, y_2) = f(x_1, y_1) \circ (x_2, y_2)$. Therefore f is a multiplier of $L_1 \times L_2$. ■

Lemma 3.17. *Let L be an ASL and f be a multiplier of L. Define $Fix_f(L) = \{x \in L \mid f(x) = x\}$. Then for any $x \in Fix_f(L)$ and $y \in L$, $x \circ y \in Fix_f(L)$.*

Proof. Let $x \in Fix_f(L)$ and $y \in L$. Now, consider $f(x \circ y) = f(x) \circ y = x \circ y$. Therefore $x \circ y \in Fix_f(L)$. ■

Corollary 3.18. *Let L be an ASL with zero and f be a multiplier of L. Then $Fix_f(L) = \{x \in L \mid f(x) = x\}$ is an ideal of L.*

Proof. Since $f(0) = 0$, $Fix_f(L)(\neq \emptyset) \subseteq L$. Then by Lemma 3.17, it is obvious that $Fix_f(L)$ is an ideal of L. ■

Theorem 3.19. *Let L be an ASL and f_1 and f_2 be multipliers of L. Then $f_1 = f_2$ if and only if $Fix_{f_1}(L) = Fix_{f_2}(L)$.*

Proof. Suppose $f_1 = f_2$. Then $f_1(x) = f_2(x)$ for all $x \in L$. It follows that, $Fix_{f_1}(L) = Fix_{f_2}(L)$. Conversely, suppose $Fix_{f_1}(L) = Fix_{f_2}(L)$. Then for any $x \in L$, $f_1(f_1(x)) = f_1(x)$. Hence $f_1(x) \in Fix_{f_1}(L) = Fix_{f_2}(L)$. Thus $f_2(f_1(x)) = f_1(x)$. Hence $f_2f_1 = f_1$. Similarly, we can show that $f_1f_2 = f_2$. Now, since f_1 and f_2 are isotone and $f_1(x) \leq x$, we get $f_2(f_1(x)) \leq f_2(x)$. Then $f_2f_1 \leq f_2$ and hence $f_1 \leq f_2$. Similarly, we can prove that $f_2 \leq f_1$. Therefore $f_1 = f_2$. ■

Theorem 3.20. *Let L be an ASL and $M(L)$ be the set of all multipliers of L. For any $f_1, f_2 \in M(L)$ if we define, $(f_1 \circ f_2)(x) = f_1(x) \circ f_2(x)$ for all $x \in L$, then $(M(L), \circ)$ is an ASL.*

Proof. Let $f_1, f_2 \in M(L)$ and $x, y \in L$. Then $(f_1 \circ f_2)(x \circ y) = f_1(x \circ y) \circ f_2(x \circ y) = (f_1(x) \circ y) \circ (f_2(x) \circ y) = (f_1(x) \circ f_2(x)) \circ y = (f_1 \circ f_2)(x) \circ y$. Therefore $f_1 \circ f_2$ is a multiplier on L and hence $M(L)$ is closed under the induced operation $\circ$ on L, and clearly, $M(L)$ satisfies all the conditions of an ASL L. Therefore $(M(L), \circ)$ is an ASL. ■

Theorem 3.21. *Let L be an ASL and $\mathcal{F} = \{Fix_f(L) \mid f \in M(L)\}$. For any $f_1,\ f_2 \in M(L)$, if we define, $Fix_{f_1}(L) \circ Fix_{f_2}(L) = Fix_{f_1 \circ f_2}(L)$, then $(\mathcal{F}, \circ)$ is an ASL and $\mathcal{F} \cong M(L)$.*

Proof. By Theorem 3.20, $\mathcal{F}$ is closed under the operation $\circ$. Since $\mathcal{F} \subseteq L$, we can clearly observe that, $(\mathcal{F}, \circ)$ is an ASL. Now, define a mapping $\phi : M(L) \longrightarrow \mathcal{F}$ by $\phi(f) = Fix_f(L)$. We know that for any $f_1,\ f_2 \in M(L)$, $Fix_{f_1}(L) = Fix_{f_2}(L)$ if and only if $f_1 = f_2$. this implies, $\phi(f_1) = \phi(f_2)$ if and only if $f_1 = f_2$. Therefore ϕ is well-defined and one-one. Now, for any $Fix_f(L) \in \mathcal{F}$, there exists $f \in M(L)$ such that $\phi(f) = Fix_f(L)$. Therefore ϕ is onto. Also, for any $f_1,\ f_2 \in M(L), \phi(f_1 \circ f_2) = Fix_{f_1 \circ f_2}(L) = Fix_{f_1}(L) \circ Fix_{f_2}(L) = \phi(f_1) \circ \phi(f_2)$. Therefore ϕ is a homomorphism and hence $\mathcal{F} \cong M(L)$. ■

Theorem 3.22. *Let f and g be two idempotent multipliers of an ASL L and their composition is commutative, i.e. $fg = gf$. Then the following are equivalent:*

(1) $f = g$.
(2) $f(L) = g(L)$.
(3) $Fix_f(L) = Fix_g(L)$.

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (3) Suppose $f(L) = g(L)$. Let $x \in Fix_f(L)$. Then $x = f(x) \in f(L) = g(L)$. Hence $x = g(y)$ for some $y \in L$. Now, $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Hence $x \in Fix_g(L)$. Therefore $Fix_f(L) \subseteq Fix_g(L)$. Similarly we can show that $Fix_g(L) \subseteq Fix_f(L)$. Therefore $Fix_f(L) = Fix_g(L)$.

(3) ⇒ (1) Suppose $Fix_f(L) = Fix_g(L)$. Let $x \in L$. Now, since $f(f(x)) = f(x)$, we have $f(x) \in Fix_f(L) = Fix_g(L)$. Hence $g(f(x)) = f(x)$. Also, $g(x) \in Fix_g(L) = Fix_f(L)$ and hence $f(g(x)) = g(x)$. Therefore $f(x) = g(f(x)) = (gf)(x) = (fg)(x) = f(g(x)) = g(x)$. Thus $f = g$. ∎

Proposition 3.23. *Let L be an ASL. Then for any $a \in L$, define $\phi_a = \{(x, y) \in L \times L \mid f_a(x) = f_a(y)\}$ where f_a is a principal multiplier of L. Then ϕ_a is a congruence relation on L.*

Proof. For any $x \in L$, it is clear that $(x, x) \in \phi_a$ and hence ϕ_a is reflexive on L. Now, let $x, y \in L$ such that $(x, y) \in \phi_a$. Then $f_a(x) = f_a(y)$ which also implies that, $a \circ x = a \circ y$. Hence $a \circ y = a \circ x$. It follows that, $f_a(y) = f_a(x)$. Therefore $(y, x) \in \phi_a$. Thus ϕ_a is symmetric on L. Clearly ϕ_a is transitive relation on L. Therefore ϕ_a is an equivalence relation on L. It remains to prove that, ϕ_a satisfies the compatibility property. Let $(x, y),\ (p, q) \in \phi_a$. Then $a \circ x = a \circ y$ and $a \circ p = a \circ q$. Consider, $a \circ x \circ p = a \circ a \circ x \circ p = a \circ x \circ a \circ p = a \circ y \circ a \circ q = a \circ y \circ q$. Hence $(x \circ p,\ \ y \circ q) \in \phi_a$. Therefore ϕ_a is a congruence relation on L. ∎

Note that, from the definition of an Almost semilattice L, for any $a,\ b,\ x \in L$, $a \circ b \circ x = b \circ a \circ x$. This implies that, $\phi_{a \circ b} = \phi_{b \circ a}$. Based on this concept, we prove that the set of all principal multipliers in an almost semilattice is a semilattice.

Theorem 3.24. *Let L be an ASL and $M(L)$ be the set of all multipliers of L. Then the set $\rho(L) = \{f_a \mid a \in L\}$, of all principal multipliers of L, is a semilattice, where $f_a \circ f_b = f_{a \circ b}$ for all $a, b \in L$.*

Proof. Let $a, b \in L$. Then $(f_a \circ f_b)(x) = f_a(x) \circ f_b(x) = (a \circ x) \circ (b \circ x) = (a \circ b) \circ x = f_{a \circ b}(x)$. Therefore $f_a \circ f_b = f_{a \circ b} \in \rho(L)$ and hence $\rho(L)$ is closed under the operation $\circ$. Thus $\rho(L)$ is a sub-ASL of L. Now, for any $x \in L$, $f_{a \circ b}(x) = (a \circ b) \circ x = (b \circ a) \circ x = f_{b \circ a}(x)$. Therefore $f_{a \circ b} = f_{b \circ a}$ and hence $f_a \circ f_b = f_b \circ f_a$. Therefore $(\rho(L), \circ)$ is a semilattice. ∎

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