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Some Properties of m-factor Set on Filippov Algebras

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Abstract. The notion of factor set in Lie algebras is defined by Moneyhun in 1994. It was developed by Eshrati et al. on n-Lie algebras. In this paper we present an m-factor set notion on Filippov algebras and obtain some results for finite dimensional m-stem Filippov algebras. Moreover, we show that if A and B are two finite dimensional m-stem Filippov algebras or have the same dimensions, then $A \sim_m B$ if and only if $A \cong B$.

Keywords: Lie algebra; Isoclinic; m-Isoclinism; n-Lie algebras.

1. Introduction and Preliminaries

The concept of n-Lie algebra was introduced by Filippov [6] in 1987. In this paper, we use Filippov algebra instead of n-Lie algebra. A Filippov algebra over a field Λ is a vector space A along with an anti-symmetric n-linear form $[x_1, \cdots, x_n]$ satisfying the Jacobi identity:

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]$$

for all $x_i, y_j \in A, 1 \leq i \leq n$ and $2 \leq j \leq n$. Clearly when $n = 2$, we have Jacobi identity in Lie algebra. A subspace B of an Filippov algebra A is called subalgebra of A if which is closed under the n-Lie product on A. Also, an n-Lie subalgebra I of A is called n-Lie ideal if, $[I, \underbrace{A, \cdots, A}_{(n-1)-time}] \subseteq I$. The center of A is defined by

$$Z(A) = \{x \in A,\ [x, y_2, \cdots, y_n] = 0, \forall y_i \in A, 2 \leq i \leq n\}$$

The lower and upper central series of A, are defined as follows

$$\cdots \subseteq A^{m+1} \subseteq A^m \subseteq \cdots \subseteq A^2 \subseteq A^1 = A$$

and

$$(0) \subseteq Z(A) = Z_1(A) \subseteq Z_2(A) \subseteq \cdots \subseteq Z_m(A) \subseteq \cdots$$

respectively, where $A^{m+1} = [A, \cdots, A, A^m]$. For $2 \leq n$ and $1 \leq m$, the subalgebra A^{m+1} is generated by elements of the form $[x_1, x_2, \cdots x_{(n-1)}, [x_n, \cdots, x_{2(n-1)}, [\cdots, [\cdots, x_{(m-1)(n-1)}, [x_{(m-1)(n-1)+1}, \cdots, x_{m(n-1)+1}]] \cdots]]]]$. Thus for convenience, put $t = (n-1)m + 1$ and use the form $[x_1, x_2, \cdots, x_t]$, where $x_1, x_2, \ldots, x_{(t-1)} \in A$ and $x_t \in A^m$. Also $Z_{m+1}(A)/Z_m(A) = Z(A/Z_m(A))$. The map $\phi : A \to B$ is called homomorphism if ϕ is an n-linear map and

$$\phi([x_1, \cdots, x_n]) = [\phi(x_1), \cdots, \phi(x_n)].$$

In 1940, the notion of m-isoclinism between two groups was presented by Bioch [1], and in 2010, introduced in Lie algebras by Salemkar and Mirzaei [13]. In [3] and [4], the authors defined the concept of m-isoclinism on Filippov algebras and studied some principle properties of m-isoclinism in the class of Filippov algebras. For further information on isoclinism (m-isoclinism) of Lie algebras and Filippov algebras, see [12, 2, 7, 8].

Now we recall the definition of m-isoclinic between two Filippov algebras.

Definition 1.1. *Let A and B be two Filippov algebras. Assume $\alpha : A/Z_m(A) \to B/Z_m(B)$ and $\beta : A^{m+1} \to B^{m+1}$ are isomorphisms such that the following diagram is commutative:*

$$\begin{array}{ccc}
\underbrace{\frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)}}_{t-times} & \longrightarrow & A^{m+1} \\
\downarrow \alpha^t & & \downarrow \beta \\
\underbrace{\frac{B}{Z_m(B)} \times \cdots \times \frac{B}{Z_m(B)}}_{t-times} & \longrightarrow & B^{m+1}
\end{array}$$

where the rule's of horizontal maps are $(\bar{x}_1, \ldots, \bar{x}_t) \longrightarrow [x_1, \ldots, x_t]$ when $x_i \in A/Z_m(A)$ and Similarly for $\overline{y_i} \in B/Z_m(B)$. Also, $\beta([x_1, \ldots, x_t]) = [y_1, \ldots, y_t]$, for all $0 \leqslant i \leqslant t$, $y_i \in \alpha(x_i + Z_m(A))$ and $t = (n-1)m + 1$. In this case the pair

(α, β) is called an m-isoclinism from A to B and they are called m- isoclinic, which is denoted by $A \sim_m B$.

In order to state our main results, we need the following lemmas, which their proofs are the same on Lie algebra (see [11, 9]).

Lemma 1.2. *Let A be an Filippov algebra and B an abelian Filippov algebra. Then $A \sim_m A \oplus B$.*

Lemma 1.3. *Let I be an ideal of Filippov algebra A. Then*

(i) *if $I \cap A^2 = 0$, then $A \sim_m A/I$,*
(ii) *if A is of finite dimension and $A \sim_m A/I$, then $I \cap A^2 = 0$.*

Lemma 1.4. *Let A and B be m-isoclinisms Filippov algebras with the given pair (α, β). Then, for each $x \in A^{m+1}$,*

(i) $\alpha(x + Z_m(A)) = \beta(x) + Z_m(B)$,
(ii) $\beta([x, x_2, \cdots, x_t] = [\beta(x), y_2, \cdots, y_t]$, *for all* $x_i \in A$, $y_i \in \alpha(x_i + Z_m(A))$,$2 \leq i \leq t$.

The concepts of stem Lie algebra, m-stem Lie algebra, and stem n-Lie algebra are defined and studied in [9], [6] and [5], respectively.

Definition 1.5. *The Filippov algebra A is said to be m-stem Filippov algebra, when $Z(A) \subseteq A^{m+1}$ for $m \geq 1$.*

In the next section, we obtain some results by using this notion.

Lemma 1.6. *Let $\mathfrak{C}$ be an isoclinism family of Filippov algebras. Then*

(i) *$\mathfrak{C}$ contains an m-stem Filippov algebra;*
(ii) *each finite dimensional Filippov algebra T in $\mathfrak{C}$ is m-stem if and only if T has minimal dimension in $\mathfrak{C}$.*

2. m-Factor Sets on Filippov Algebras

The notion of factor sets in Lie algebra is defined by Moneyhun [10] in 1994, and he showed that if L and M are two finite dimensional stem Lie algebras or they have the same dimensions, then $L \sim M$ if and only if $L \cong M$. The same notions for n-Lie algebras studied by Eshrati et al. in [5]. In this section, we give a more general definition of the concept of m-factor set in Filippov algebras, which includes the previous ones and gives some of its properties. We show that, if A and B are two finite dimensional m-stem Filippov algebras or have the same dimensions, then $A \sim_m B$ if and only if $A \cong B$.

Definition 2.1. *Let A be a finite dimensional Filippov algebra over a field Λ. For each $1 \leq m$ and $t = (n-1)m + 1$, the t-linear map*

$$f_m : \frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)} \longrightarrow Z_m(A)$$

is called an m-factor set if it satisfies the following conditions:

(i) $f_m(\overline{x}_1, \ldots, \overline{x}_i, \ldots, \overline{x}_j, \ldots, \overline{x}_t) = 0$, *if* $\overline{x}_i = \overline{x}_j$, *for all* $\overline{x}_k = x_k + Z_m(A) \in A/Z_m(A)$.

(ii) $f_m([\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t) = \sum_{i=1}^{t} f_m(\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t)$, *for all* $\overline{x}_i, \overline{y}_j \in A/Z_m(A)$, $1 \leq i \leq t$ *and* $2 \leq j \leq t$.

The following lemmas are needed for the proofs of our main results.

Lemma 2.2. *Let A be a Filippov algebra and f_m be an m-factor set on A. Then*

(i) *the set*

$$R = (Z_m(A), \frac{A}{Z_m(A)}, f_m) = \{(a, \overline{x}) : a \in Z_m(A), \overline{x} \in \frac{A}{Z_m(A)}\},$$

is a Filippov algebra under the following multiplication

$$[(a_1, \overline{x}_1), \ldots, (a_t, \overline{x}_t)] = (f_m(\overline{x}_1, \ldots, \overline{x}_t), [\overline{x}_1, \ldots, \overline{x}_t]).$$

(ii) $Z_R = \{(a, 0) \in R : a \in Z_m(A)\} \cong Z_m(A)$.

Proof. (i) Clearly, R is an algebra and so one only need to check the properties of Filippov algebra. Hence, if $(a_i, \overline{x}_i) = (a_j, \overline{x}_j)$, then

$$[(a_1, \overline{x}_1), \ldots, (a_i, \overline{x}_i), \ldots, (a_j, \overline{x}_j), \ldots, (a_t, \overline{x}_t)] = 0,$$

for all $a_k \in Z_m(A)$, $\overline{x}_k \in A/Z_m(A)$ and $1 \leq k \leq t$. Now we show the Jacobi identity,

$$\begin{aligned}
& [[(a_1, \overline{x}_1), \ldots, (a_t, \overline{x}_t)], (b_2, \overline{y}_2), \ldots, (b_t, \overline{y}_t)] \\
= \; & [(f_m(\overline{x}_1, \ldots, \overline{x}_t), [\overline{x}_1, \ldots, \overline{x}_t]), (b_2, \overline{y}_2), \ldots, (b_t, \overline{y}_t)] \\
= \; & (f_m([\overline{x}_1, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t), [[\overline{x}_1, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t]) \\
= \; & (\sum_{i=1}^{t} f_m(\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t), \sum_{i=1}^{t} [\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t]) \\
= \; & \sum_{i=1}^{n} (f_m(\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t), [\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t])
\end{aligned}$$

$$= \sum_{i=1}^{t} [(a_1, \overline{x}_1), \ldots, (f_m(\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t), [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t]), \ldots, (a_t, \overline{x}_t)]$$
$$= \sum_{i=1}^{t} [(a_1, \overline{x}_1), \ldots, [(a_i, \overline{x}_i), (b_2, \overline{y}_2), \ldots, (b_t, \overline{y}_t)], \ldots, (a_t, \overline{x}_t)] \, .$$

Hence, R is a Filippov algebra.

(ii) Define the map $\phi : Z_m(A) \to Z_R$, given by $\phi(z) = (z, 0)$, for all $z \in Z_m(A)$. Clearly ϕ is an isomorphism and so $Z_R \cong Z_m(A)$. ■

The following lemma shows that every Filippov algebra has an m-factor set.

Lemma 2.3. *Let A be a Filippov algebra. Then there exists an m-factor set f_m on A such that*
$$A \cong (Z_m(A), \frac{A}{Z_m(A)}, f_m).$$

Proof. Put K to be a complement of $Z_m(A)$ in A, hence $A = K \oplus Z_m(A)$. Now define the map $T : A/Z_m(A) \longrightarrow A$, given by $T(\overline{x}) = T(x + Z_m(A)) = T(k + a + Z_m(A)) = k$, for all $x \in A$, $a \in Z_m(A)$ and $k \in K$. Clearly, $\overline{T(\overline{x})} = \overline{x}$ and
$$[T(\overline{x}_1), T(\overline{x}_2), \ldots, T(\overline{x}_t)] - T[\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t] \in Z_m(A).$$

Consider the map
$$f_m : \underbrace{\frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)}}_{t-time} \longrightarrow Z_m(A)$$

defined by
$$f_m(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_t) = [T(\overline{x}_1), T(\overline{x}_2), \cdots, T(\overline{x}_t)] - T[\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_t],$$

so that f_m is an m-factor set. Part (i) of Definition 2.1 is clear. So for part (ii) we check Jacobi identity. Hence

$$\begin{aligned}
& f_m([\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t) \\
&= [T([\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t]), T(\overline{y}_2), \ldots, T(\overline{y}_t)] - T([[\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t]) \\
&= [[T(\overline{x}_1), T(\overline{x}_2), \ldots, T(\overline{x}_t)], T(\overline{y}_2), \ldots, T(\overline{y}_t)] - T([[\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t], \overline{y}_2, \ldots, \overline{y}_t]) \\
&= \sum_{i=1}^{t} [T(\overline{x}_1), \ldots, [T(\overline{x}_i), T(\overline{y}_2), \ldots, T(\overline{y}_t)], \ldots, T(\overline{x}_t)] \\
&\quad - T(\sum_{i=1}^{t} [\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t])
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{t} [T(\overline{x}_1), \ldots, T([\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t]), \ldots, T(\overline{x}_t)] \\
&\quad - \sum_{i=1}^{t} T\left([\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t]\right) \\
&= \sum_{i=1}^{t} ([T(\overline{x}_1), \ldots, T([\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t]), \ldots, T(\overline{x}_t)]) \\
&\quad - T([\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t]) \\
&= \sum_{i=1}^{t} f_m(\overline{x}_1, \ldots, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_t], \ldots, \overline{x}_t).
\end{aligned}
$$

Now, define the map $\phi : (Z_m(A), A/Z_m(A), f_m) \longrightarrow A$ given by $\phi(a, \overline{x}) = a + T(\overline{x})$, for all $a \in Z_m(A)$ and $\overline{x} = x + Z_m(A) = k + Z_m(A) \in A/Z_m(A)$. Clearly, ϕ is a well-defined map and it is isomorphism. ■

The relationship between the two m-stem Filippov algebras is shown in the following lemma.

Lemma 2.4. *Let A be an m-stem Filippov algebra in m-isoclinism family of Filippov algebras $\mathfrak{C}$. Then for any m-stem Filippov algebra B of $\mathfrak{C}$, there exists an m-factor set f_m on A, such that:*

$$B \cong (Z_m(A), \frac{A}{Z_m(A)}, f_m)$$

Proof. Let (α, β) be a pair of m-isoclinism between Filippov algebras A and B. Then by Lemma 1.2, $\beta(Z_m(A)) = Z_m(B)$. According to Lemma 2.3, there exists an m-factor set g_m such that

$$B \cong (Z_m(B), \frac{B}{Z_m(B)}, g_m).$$

Now, define the m-factor set

$$f_m : \underbrace{\frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)}}_{t-time} \longrightarrow Z_m(A)$$

given by

$$f_m(\overline{x}_1, \ldots, \overline{x}_t) = \beta^{-1}(g_m(\alpha(\overline{x}_1)), \ldots, \alpha(\overline{x}_t))),$$

for all $\overline{x}_1, \ldots, \overline{x}_t \in A/Z_m(A)$. Let

$$\varphi : (Z_m(A), \frac{A}{Z_m(A)}, f_m) \longrightarrow (Z_m(B), \frac{B}{Z_m(B)}, g_m)$$

be given by $\varphi(a,\overline{x}) = (\beta(a),\alpha(\overline{x}))$, for all $a \in Z_m(A)$ and $\overline{x} \in A/Z_m(A)$. It is easy to see that the map φ is a well-defined bijection and also we have

$$\begin{aligned}
\varphi[(a_1,\overline{x}_1),\ldots,(a_t,\overline{x}_t)] &= \varphi(f_m(\overline{x}_1,\ldots,\overline{x}_t),[\overline{x}_1,\ldots,\overline{x}_t]) \\
&= (\beta(f_m(\overline{x}_1,\ldots,\overline{x}_t)),\alpha([\overline{x}_1,\ldots,\overline{x}_t])) \\
&= (g_m(\alpha(\overline{x}_1),\ldots,\alpha(\overline{x}_t)),\alpha([\overline{x}_1,\ldots,\overline{x}_t])) \\
&= [(\beta(a_1),\alpha(\overline{x}_1)),\ldots,(\beta(a_t),\alpha(\overline{x}_t))] \\
&= [\varphi(a_1,\overline{x}_1),\ldots,\varphi(a_t,\overline{x}_t)].
\end{aligned}$$

So φ is an isomorphism of Filippov algebras and the proof is completed. ■

Lemma 2.5. *Let A be a Filippov algebra and f_m and g_m be two m-factor sets on A. Assume that*

$$\begin{aligned}
R &= (Z_m(A),\frac{A}{Z_m(A)},f_m), \quad Z_R = \{(a,0) \in R : a \in Z_m(A)\}, \\
S &= (Z_m(A),\frac{A}{Z_m(A)},g_m), \quad Z_S = \{(a,0) \in S : a \in Z_m(A)\}.
\end{aligned}$$

If λ is an isomorphism from R onto S satisfying $\lambda(Z_R) = Z_S$, then the restriction of λ on $A/Z_m(A)$ and $Z_m(A)$ define the automorphisms $\alpha \in Aut(A/Z_m(A))$ and $\beta \in Aut(Z_m(A))$, respectively.

Proof. See [5, Lemma 2.7] for the proof. ■

Lemma 2.6. *Let A be a Filippov algebra, R, S, Z_R, and Z_S be as in the previous lemma. Then the following statements hold:*

(i) *If $\lambda : R \to S$ is a Filippov isomorphism such that $\lambda(Z_R) = Z_S$ and α,β are the corresponding automorphisms induced by λ on $A/Z_m(A)$ and $Z_m(A)$, respectively. Then there exists a linear map $\gamma : A/Z_m(A) \longrightarrow Z_m(A)$ such that*

$$\beta(f_m(\overline{x}_1,\ldots,\overline{x}_t)) + \gamma([\overline{x}_1,\ldots,\overline{x}_t]) = g_m(\alpha(\overline{x}_1),\ldots,\alpha(\overline{x}_t)).$$

(ii) *If $\alpha \in Aut(A/Z_m(A))$, $\beta \in Aut(Z_m(A))$, and $\delta : A/Z_m(A) \longrightarrow Z_m(A)$ are linear maps satisfying*

$$\beta(f_m(\overline{x}_1,\ldots,\overline{x}_t)) + \delta([\overline{x}_1,\ldots,\overline{x}_t]) = g_m(\alpha(\overline{x}_1),\ldots,\alpha(\overline{x}_t)),$$

then $R \cong S$ via the isomorphism λ induced by α and β satisfying $\lambda(Z_R) = Z_S$.

Proof. The proof is similar to Lemma 2.6 in [5]. ■

Now we are in a position to state and prove our main results.

Theorem 2.7. *Let A and B be two finite dimensional m-stem Filippov algebras. Then $A \sim_m B$ if and only if $A \cong B$.*

Proof. The proof of sufficient condition is obvious. To prove the other side assume that $A \sim_m B$. According to Lemma 2.4, there exist m-factor sets f_m and g_m such that

$$A \cong (Z_m(A), \frac{A}{Z_m(A)}, f_m) = R \quad \text{and} \quad B \cong (Z_m(A), \frac{A}{Z_m(A)}, g_m) = S$$

.Now suppose the pair (ω, τ) is an m-isoclinic between two Filippov algebras R and S. Then $Z_m(R) \cong Z_m(A) \cong Z_R$ and $Z_m(S) \cong Z_m(B) \cong Z_S$. As $Z_R \subseteq Z_m(R)$, we have $Z_m(R) = Z_R$. Now, let $\alpha \in Aut(A/Z_m(A))$ be a map defined by

$$\omega((0, \overline{x}) + Z_R) = (0, \alpha(\overline{x})) + Z_S,$$

for all $\overline{x} \in A/Z_m(A)$. Also suppose $\beta \in Aut(Z_m(A))$ defined by $\tau(a, 0) = (\beta(a), 0)$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)} & \xrightarrow{\rho} & \frac{R}{Z_R} \times \cdots \times \frac{R}{Z_R} & \xrightarrow{\theta} & R^{m+1} \\
\downarrow \alpha^t & & \downarrow \omega^t & & \downarrow \tau \\
\frac{A}{Z_m(A)} \times \cdots \times \frac{A}{Z_m(A)} & \xrightarrow[\sigma]{} & \frac{S}{Z_S} \times \cdots \times \frac{S}{Z_S} & \xrightarrow{\xi} & S^{m+1},
\end{array}$$

where

$$\begin{aligned}
\rho(\overline{x}_1, \ldots, \overline{x}_t) &= ((0, \overline{x}_1) + Z_R, \ldots, (0, \overline{x}_t) + Z_R), \\
\sigma(\overline{x}_1, \ldots, \overline{x}_t) &= ((0, \overline{x}_1) + Z_S, \ldots, (0, \overline{x}_t) + Z_S), \\
\theta((a_1, \overline{x}_1) + Z_R, \ldots, (a_t, \overline{x}_t) + Z_R) &= [(a_1, \overline{x}_1), \ldots, (a_t, \overline{x}_t)] \\
&= (f_m(\overline{x}_1, \ldots, \overline{x}_t), [\overline{x}_1, \ldots, \overline{x}_t], \\
\xi((a_1, \overline{x}_1) + Z_S, \ldots, (a_t, \overline{x}_t) + Z_S) &= [(a_1, \overline{x}_1), \ldots, (a_t, \overline{x}_t)] \\
&= (g_m(\overline{x}_1, \ldots, \overline{x}_t), [\overline{x}_1, \ldots, \overline{x}_t].
\end{aligned}$$

Using the right hand side of the above diagram and the property $R \sim_m S$, we have

$$\begin{aligned}
\tau[(0, \overline{x}_1), \ldots, (0, \overline{x}_t)] &= [(0, .(\overline{x}_1)), \ldots, (0, .(\overline{x}_t))] \\
&= (g_m(.(\overline{x}_1), \ldots, .(\overline{x}_t)), [.(\overline{x}_1), \ldots, .(\overline{x}_t)]) \qquad (1)
\end{aligned}$$

On the other hand, by assuming $\delta : A^{m+1}/Z_m(A) \longrightarrow Z_m(A)$ given by

$$\begin{aligned}
\tau[(0, \overline{x}_1), \ldots, (0, \overline{x}_t)] &= \tau(f_m(\overline{x}_1, \ldots, \overline{x}_t), [\overline{x}_1, \ldots, \overline{x}_t]) \\
&= \tau(f_m(\overline{x}_1, \ldots, \overline{x}_t), 0) + \tau(0, [\overline{x}_1, \ldots, \overline{x}_t]) \\
&= (\beta(f_m(\overline{x}_1, \ldots, \overline{x}_t)), 0) + (\delta[\overline{x}_1, \ldots, \overline{x}_t], *) \\
&= (\beta(f_m(\overline{x}_1, \ldots, \overline{x}_t)) + \delta[\overline{x}_1, \ldots, \overline{x}_t], *), \qquad (2)
\end{aligned}$$

where $*$ is element of $A/Z_m(A)$. The Eqs. (1) and (2) give the following equation

$$\beta(f_m(\overline{x}_1, \cdots, \overline{x}_t)) + \delta[\overline{x}_1, \cdots, \overline{x}_t] = g_m(\alpha(\overline{x}_1), \cdots, \alpha(\overline{x}_t)).$$

Now we extend the linear map δ on $A/Z_m(A)$ to obtain the conditions of Lemma 2.6 (ii). Assume that it is a pure subalgebra in the complement of $A^{m+1}/Z_m(A)$ in $A/Z_m(A)$. Then we obtain $R \cong S$. ∎

Theorem 2.8. *Let $\mathfrak{C}$ be an m-isoclinism family of Filippov algebras of finite dimension. Then $\mathfrak{C}$ possesses a stem Filippov algebra T. Also any Filippov algebra A in $\mathfrak{C}$ can be expressed as the direct sum of T with some finite dimensional abelian Filippov algebra.*

Proof. By Lemma 1.6, $\mathfrak{C}$ contains an m-stem Filippov algebra T. By Lemma 1.2, $T \sim_m T \oplus A$, for any abelian Filippov algebra A, and so $T \oplus A$ is in $\mathfrak{C}$. Let A be arbitrary Filippov algebra in $\mathfrak{C}$ and S be the complement of $Z_m(A) \cap A^{m+1}$ in $Z_m(A)$, that is,

$$S \oplus (Z_m(A) \cap A^{m+1}) = Z_m(A).$$

As $S \subseteq Z_m(A)$, we have $[S, A, \cdots, A] \subseteq [Z_m(A), A, \cdots, A]$ and so S is an ideal of A. Assume $T = A/S$. Then $S \cap (Z_m(A) \cap A^{m+1}) = S \cap A^{m+1} = 0$ and by Lemma 1.3, $A/S \sim_m A$. Also,

$$Z_m(T) = Z_m\left(\frac{A}{S}\right) = \frac{Z_m(A)}{S} \subseteq \frac{A^{m+1} + S}{S} \cong \left(\frac{A}{S}\right)^{m+1} = T^{m+1}.$$

Therefore T is m-stem Filippov algebra.

Now, there exists a subspace K of A such that $A^{m+1} \subseteq K, k \cap s = 0$ and $K \oplus S = A$. Hence $[K, A, \cdots, A] \subseteq [A, A, \ldots, A] = A^{m+1} \subseteq K$ and consequently $K \lhd A$. Also, $A \sim_m A/S \cong (K \oplus S)/S \cong K$, and so K is a finite dimensional m-stem Filippov algebra. Clearly $T \sim_m A \sim_m K$, which gives $T \sim_m K$ and hence $T \cong K$. In particular, $A = K \oplus S \cong T \oplus S$. ∎

Theorem 2.9. *Let A and B be two Filippov algebras with the same dimensions. Then $A \sim_m B$ if and only if $A \cong B$.*

Proof. Suppose that $A \sim_m B$. Then $A = T \oplus S_1$ and $B = T' \oplus S_2$, where T and T' are m-stem Filippov algebras. By Theorem 2.7, $T \cong T'$. Now, since $S_1 \cong S_2$, it follows that $T \oplus S_1 \cong T' \oplus S_2$. Therefore $A \cong B$. ∎

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