Southeast Asian Bulletin of Mathematics (2024) 48: 195–210

Southeast Asian
Bulletin of
Mathematics

A Class of Nonlinear Non-global Semi-Jordan Triple Higher Derivable Mappings on Triangular Algebras*

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Received 13 June 2022
Accepted 19 September 2023

Communicated by Y.Q. Chen

AMS Mathematics Subject Classification(2020): 16W25, 47L35

Abstract. In this paper, we proved that each nonlinear non-global semi-Jordan triple higher derivable mapping on a 2-torsion free triangular algebra is an additive higher derivation. As its application, we get the similar conclusion on a nest algebra or a 2-torsion free block upper triangular matrix algebra, respectively.

Keywords: Semi-Jordan triple derivable mapping; Semi-Jordan triple higher derivable mapping; Derivation; Higher derivation.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ a unital algebra over $\mathcal{R}$, $\Omega = \{X \in \mathcal{A} : X^2 = 0\}$, $\mathbb{N}$ be the set of non-negative integers, $i, j, k, n \in \mathbb{N}$, Δ be an additive mapping on $\mathcal{A}$, $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence additive mapping on $\mathcal{A}$ ($d_0 = id_{\mathcal{A}}$ the identity mapping). For any $X, Y \in \mathcal{A}$, denote the Jordan product of X, Y by $X \circ Y = XY + YX$. For any $X \in \mathcal{A}$, if $2X = 0$, implies $X = 0$, then $\mathcal{A}$ is said to be a 2-torsion free algebra. Recall that Δ is called a derivation (resp. Jordan derivation) if $\Delta(XY) = \Delta(X)Y + X\Delta(Y)$ (resp. $\Delta(X \circ Y) = \Delta(X) \circ Y + X \circ \Delta(Y)$) for all $X, Y \in \mathcal{A}$; D is said to be a higher

*The research is supported by Talent Project Foundation of Yunnan Province (No.202105AC160089), Natural Science Foundation of Yunnan Province (No.202101BA070001198), and Basic Research Foundation of Yunnan Education Department(Nos. 2020J0748 and 2021J0915).

derivation (resp. Jordan higher derivation) if $d_n(XY) = \sum_{i+j=n} d_i(X)d_j(Y)$ (resp. $d_n(X \circ Y) = \sum_{i+j=n} d_i(X) \circ d_j(Y)$) for all $n \in \mathbb{N}, X, Y \in \mathcal{A}$. Furthermore, if Δ and D without assumption of additivity in the above definitions, then Δ is said to be a nonlinear (Jordan) derivable mapping, and D is called a nonlinear (Jordan) higher derivable mapping, respectively. Obviously, every derivation is a Jordan derivation and every higher derivation is a Jordan higher derivation. However, the inverse statement is not true in general.

An important and interesting problem that we want to know what conditions can imply a Jordan (higher) derivation (resp. nonlinear Jordan (higher) derivable mapping) is a (higher) derivation, respectively.

In the past 60 years, many mathematicians studied this problem and obtained abundant results. For example, each Jordan derivation on a prime or semi-prime ring not of characteristic 2 is a derivation (see [3], [5] and [10]). Every Jordan derivation on a nest algebra or a 2-torsion free triangular algebra is a inner derivation or a derivation (see [16, 17]), respectively. Hoger in [9] extended the result of Zhang in [17] and proved that under certain conditions, each Jordan derivation on trivial extension algebras is a sum of a derivation and an anti-derivation. Other similar results about Jordan derivations (nonlinear Jordan derivable mappings), we refer the readers to [3], [5], [10], [13] and references therein for more details.

With the deepening of research, the research of Jordan higher derivations (nonlinear Jordan higher derivable mappings) has also attracted extensive attention of many scholars and many research results have been achieved . For example, Xiao and Wei in [15] obtained that every Jordan higher derivation on triangular algebras is a higher derivation. This result was extended by Fu and Xiao in [8]. They proved that each nonlinear Jordan higher derivable mapping on triangular algebras is a higher derivation. H.R.E. Vishki et al. in [14] proved that under certain conditions each Jordan higher derivation on a trivial extension algebra is a higher derivation. Other similar results about Jordan higher derivations (nonlinear Jordan higher derivable mappings), are presente in [11], [12], [18] and references therein for more details. In particularly, M. Ashraf and A. Jabeen in [1] proved that if $D = \{d_n\}_{n\in\mathbb{N}}$ without the additivity assumption and satisfies

$$d_n(XYZ + ZYX) = \sum_{i+j+k=n} \{d_i(X)d_j(Y)d_k(Z) + d_i(Z)d_j(Y)d_k(X)\}$$

for all $n \in \mathbb{N}, X, Y, Z \in \mathcal{A}$, then such a D on a 2-torsion free triangular algebra is an additive higher derivation.

In [7], we call Δ a nonlinear non-global semi-Jordan triple derivable mapping on $\mathcal{A}$, if Δ without the additivity assumption and satisfies

$$\begin{aligned}\Delta(XYZ + YXZ) = {} & \Delta(X)YZ + X\Delta(Y)Z + XY\Delta(Z) + \Delta(Y)XZ \\ & + Y\Delta(X)Z + YX\Delta(Z)\end{aligned}$$

for all $X, Y, Z \in \mathcal{A}$ with $XYZ \in \Omega$. Furthermore, we proved that such a Δ on a 2-torsion free triangular algebra is an additive derivation (see [7, Theorem 1]).

In this paper, we say $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear non-global semi-Jordan triple higher derivable mapping on $\mathcal{A}$, if $D = \{d_n\}_{n\in\mathbb{N}}$ without the additivity assumption and satisfies

$$d_n(XYZ + YXZ) = \sum_{i+j+k=n} \{d_i(X)d_j(Y)d_k(Z) + d_i(Y)d_j(X)d_k(Z)\}$$

for all $n \in \mathbb{N}, X, Y, Z \in \mathcal{A}$ with $XYZ \in \Omega$. Here, it needs to be pointed out that our above definition is different to M. Ashraf's and A. Jabeen's in [1]. The main purpose of this paper is that extend the result of [7, Theorem 1] for semi-Jordan triple higher derivable mapping on triangular algebras.

For the convenience of reading, we give some basic concepts and properties of triangular algebras as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. Then the $\mathcal{R}$-algebra

$$\mathcal{U} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. We refer the reader to [4] for more details about the triangular algebras. Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras.

Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. Throughout this paper, we shall use the following notations

$$P_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}.$$

It is clear that the triangular algebra $\mathcal{U}$ may be represented as

$$\mathcal{U} = P_1\mathcal{U}P_1 + P_1\mathcal{U}P_2 + P_2\mathcal{U}P_2 = \mathcal{A} + \mathcal{M} + \mathcal{B}.$$

Here $P_1\mathcal{U}P_1$ and $P_2\mathcal{U}P_2$ are subalgebras of $\mathcal{U}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $P_1\mathcal{U}P_2$ is a $(P_1\mathcal{U}P_1, P_2\mathcal{U}P_2)$-bimodule isomorphic to the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$.

2. Nonlinear Non-Global Semi-Jordan Triple Higher Derivable Mappings on Triangular Algebras

Theorem 1. *Let $\mathcal{U}$ be a 2-torsion free triangular algebra and $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence mappings from $\mathcal{U}$ into itself (without assumption of additivity) such that*

$$d_n(XYZ + YXZ) = \sum_{i+j+k=n} \{d_i(X)d_j(Y)d_k(Z) + d_i(Y)d_j(X)d_k(Z)\} \tag{1}$$

for any $n \in \mathbb{N}, X, Y, Z \in \mathcal{U}$ *with* $XYZ \in \Omega$. *Then* D *is an additive higher derivation on* $\mathcal{U}$.

In fact, Theorem 1 in this paper is a generalization of [7, Theorem 1]. To prove Theorem 1, we will introduce Lemmas 2–6, and then prove that Lemmas 2–6 hold by using the mathematical induction. We assume that $\mathcal{U}$ is a 2-torsion free triangular algebra, $\Omega = \{X \in \mathcal{U} : X^2 = 0\}$, and $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear non-global semi-Jordan triple higher derivable mapping on triangular algebra $\mathcal{U}$. Let $\mathbb{N}$ be the set of non-negative integers, $\mathbb{N}^+$ be the set of positive integers, and $i, j, k, p, q, n \in \mathbb{N}$. It is known from [7, Theorem 1] that d_1 is an additive derivation on $\mathcal{U}$, and d_1 satisfies the following properties (P1):

(i) $d_1(0) = 0, d_1(P_1) = -d_1(P_2) \in \mathcal{M}$;
(ii) $d_1(X_{11}) \in \mathcal{A} + \mathcal{M}(\forall X_{11} \in \mathcal{A})$;
(iii) $d_1(X_{22}) \in \mathcal{M} + \mathcal{B}(\forall X_{22} \in \mathcal{B})$;
(iv) $d_1(X_{12}) \in \mathcal{M}(\forall X_{12} \in \mathcal{M})$;
(v) $d_1(XY) = \sum_{i+j=1} d_i(X)d_j(Y)(\forall X, Y \in \mathcal{U})$.

Now, we assume that d_k $(1 \leqslant k < n)$ have the additivity on $\mathcal{U}$ and satisfy the following properties (P2):

(i) $d_k(0) = 0, d_1(P_k) = -d_k(P_2) \in \mathcal{M}$;
(ii) $d_k(X_{11}) \in \mathcal{A} + \mathcal{M}(\forall X_{11} \in \mathcal{A})$;
(iii) $d_k(X_{22}) \in \mathcal{M} + \mathcal{B}(\forall X_{22} \in \mathcal{B})$;
(iv) $d_k(X_{12}) \in \mathcal{M}(\forall X_{12} \in \mathcal{M})$;
(v) $d_k(XY) = \sum_{i+j=k} d_i(X)d_j(Y)(\forall X, Y \in \mathcal{U})$.

Lemma 2. *For any* $n \in \mathbb{N}^+$ *and* $X_{12} \in \mathcal{M}$, *we have* $d_n(0) = 0$, $d_n(P_1) = -d_n(P_2) \in \mathcal{M}$ *and* $d_n(X_{12}) \in \mathcal{M}$.

Proof. For any $n \in \mathbb{N}^+$, taking $X = Y = Z = 0$ in Eq. (1), it follows from the properties (P2) and the property of 2-torsion free of $\mathcal{U}$ that

$$\begin{aligned} d_n(0) &= \sum_{i+j+k=n} 2d_i(0)d_j(0)d_k(0) \\ &= 2\sum_{i+j+k=n, 1\leq i,j,k} d_i(0)d_j(0)d_k(0) + 2\sum_{j+k=n, 1\leq j,k} 0d_j(0)d_k(0) \\ &\quad +2\sum_{i+k=n, 1\leq i,k} d_i(0)0d_k(0) + 2\sum_{i+j=n, 1\leq i,j} d_i(0)d_j(0)0 \\ &\quad +\{d_n(0)0 + 0d_n(0)0 + 0d_n(0)\} \\ &= 0. \end{aligned}$$

Since $P_1P_1P_2 = 0 \in \Omega$, we take $X = P_1, Y = P_1, Z = P_2$ in Eq. (1). Then by the properties (P2), we get

$$\begin{aligned} 0 &= d_n(P_1P_1P_2 + P_1P_1P_2) \\ &= 2\sum_{i+j+k=n} d_i(P_1)d_j(P_1)d_k(P_2) \\ &= 2\sum_{i+j+k=n,1\le i,j,k} d_i(P_1)d_j(P_1)d_k(P_2) + 2\sum_{j+k=n,1\le j,k} P_1d_j(P_1)d_k(P_2) \\ &\quad +2\sum_{i+k=n,1\le i,k} d_i(P_1)P_1d_k(P_2) + 2\sum_{i+j=n,1\le i,j} d_i(P_1)d_j(P_1)P_2 \\ &\quad +2d_n(P_1)P_1P_2 + 2P_1d_n(P_1)P_2 + 2P_1P_1d_n(P_2) \\ &= 2P_1d_n(P_1)P_2 + 2P_1d_n(P_2). \end{aligned}$$

And then we get from the property of 2-torsion free of $\mathcal{U}$ that

$$P_1d_n(P_2)P_1 = 0 \text{ and } P_1d_n(P_1)P_2 + P_1d_n(P_2)P_2 = 0. \tag{2}$$

Similarly, we get that

$$P_2d_n(P_1)P_2 = 0. \tag{3}$$

For any $n \in \mathbb{N}^+$ and $X_{12} \in \mathcal{M}$, since $P_2X_{12}P_2 = 0 \in \Omega$, we take $X = P_2, Y = X_{12}, Z = P_2$ in Eq. (1). Then we can get from the properties (P2) and the property of 2-torsion free of $\mathcal{U}$ that

$$\begin{aligned} d_n(X_{12}) &= d_n(P_2X_{12}P_2 + X_{12}P_2P_2) \\ &= \sum_{i+j+k=n} \{d_i(P_2)d_j(X_{12})d_k(P_2) + d_i(X_{12})d_j(P_2)d_k(P_2)\} \\ &= \sum_{i+j+k=n,1\le i,j,k} \{d_i(P_2)d_j(X_{12})d_k(P_2) + d_i(X_{12})d_j(P_2)d_k(P_2)\} \\ &\quad + \sum_{j+k=n,1\le j,k} \{P_2d_j(X_{12})d_k(P_2) + X_{12}d_j(P_2)d_k(P_2)\} \\ &\quad + \sum_{i+k=n,1\le i,k} \{d_i(P_2)X_{12}d_k(P_2) + d_i(X_{12})P_2d_k(P_2)\} \\ &\quad + \sum_{i+j=n,1\le i,j} \{d_i(P_2)d_j(X_{12})P_2 + d_i(X_{12})d_j(P_2)P_2\} \\ &\quad +d_n(P_2)X_{12}P_2 + d_n(X_{12})P_2 + P_2d_n(X_{12})P_2 + X_{12}d_n(P_2)P_2 \\ &\quad +P_2X_{12}d_n(P_2) + X_{12}P_2d_n(P_2) \\ &= d_n(P_2)X_{12} + d_n(X_{12})P_2 + P_2d_n(X_{12})P_2 + 2X_{12}d_n(P_2)P_2. \end{aligned}$$

And so we get from $P_1d_n(P_2)P_1 = 0$, the property of 2-torsion free of $\mathcal{U}$ and the faithfulness of $\mathcal{M}$ that

$$P_1d_n(X_{12})P_1 = P_2d_n(X_{12})P_2 = P_2d_n(P_2)P_2 = 0. \tag{4}$$

Similarly, we can get that

$$P_1 d_n(P_1)P_1 = 0. \tag{5}$$

Therefore, by the Eqs. (2)-(5), we get $d_n(P_1) = -d_n(P_2) \in \mathcal{M}$ and $d_n(X_{12}) \in \mathcal{M}$. The proof is completed. ∎

Lemma 3. *For any $n \in \mathbb{N}^+, X_{11} \in \mathcal{A}$ and $X_{22} \in \mathcal{B}$, we have $d_n(X_{11}) \in \mathcal{A}+\mathcal{M}$ and $d_n(X_{22}) \in \mathcal{M}+\mathcal{B}$.*

Proof. For any $n \in \mathbb{N}^+$ and $X_{11} \in \mathcal{A}$, since $X_{11}P_2P_2 = 0 \in \Omega$, we take $X = X_{11}$, $Y = Z = P_2$ in Eq. (1). Then we can get from the properties (P2) and Lemma 2 that

$$\begin{aligned}
0 &= d_n(X_{11}P_2P_2 + P_2X_{11}P_2)\\
&= \sum_{i+j+k=n} \{d_i(X_{11})d_j(P_2)d_k(P_2) + d_i(P_2)d_j(X_{11})d_k(P_2)\}\\
&= \sum_{i+j+k=n,1\leq i,j,k} \{d_i(X_{11})d_j(P_2)d_k(P_2) + d_i(P_2)d_j(X_{11})d_k(P_2)\}\\
&\quad + \sum_{j+k=n,1\leq j,k} \{X_{11}d_j(P_2)d_k(P_2) + P_2d_j(X_{11})d_k(P_2)\}\\
&\quad + \sum_{i+k=n,1\leq i,k} \{d_i(X_{11})P_2d_k(P_2) + d_i(P_2)X_{11}d_k(P_2)\}\\
&\quad + \sum_{i+j=n,1\leq i,j} \{d_i(X_{11})d_j(P_2)P_2 + d_i(P_2)d_j(X_{11})P_2\}\\
&\quad + d_n(X_{11})P_2 + X_{11}d_n(P_2)P_2 + P_2d_n(X_{11})P_2\\
&= \sum_{i+j=n,1\leq i,j} d_i(X_{11})d_j(P_2)P_2\\
&\quad + d_n(X_{11})P_2 + X_{11}d_n(P_2)P_2 + P_2d_n(X_{11})P_2.
\end{aligned}$$

This implies that $P_2d_n(X_{11})P_2 = 0$. Similarly, for any $X_{22} \in \mathcal{B}$, we can get that $P_1d_n(X_{22})P_1 = 0$. The proof is completed. ∎

Lemma 4. *For any $n \in \mathbb{N}^+$, $X_{11}, Y_{11} \in \mathcal{A}$, $X_{12} \in \mathcal{M}$ and $X_{22}, Y_{22} \in \mathcal{B}$, the following statements hold:*

(i) $d_n(X_{11}X_{22}) = \sum_{i+j=n} d_i(X_{11})d_j(X_{22}) = 0$;
(ii) $d_n(X_{11}X_{12}) = \sum_{i+j=n} d_i(X_{11})d_j(X_{12})$;
(iii) $d_n(X_{12}X_{22}) = \sum_{i+j=n} d_i(X_{12})d_j(X_{22})$;
(iv) $d_n(X_{11}Y_{11}) = \sum_{i+j=n} d_i(X_{11})d_j(Y_{11})$;
(v) $d_n(X_{22}Y_{22}) = \sum_{i+j=n} d_i(X_{22})d_j(Y_{22})$.

Proof. (i) For any $n \in \mathbb{N}^+$, $X_{11} \in \mathcal{A}$ and $X_{22} \in \mathcal{B}$, since $X_{11}X_{22}P_2 = 0 \in \Omega$, we take $X = X_{11}, Y = X_{22}, Z = P_2$ in Eq. (1). Then we can get from the properties

(P2), Lemmas 2 and 3 that

$$\begin{aligned}
0 &= d_n(X_{11}X_{22}) \\
&= d_n(X_{11}X_{22}P_2 + X_{22}X_{11}P_2) \\
&= \sum_{i+j+k=n} \{d_i(X_{11})d_j(X_{22})d_k(P_2) + d_i(X_{22})d_j(X_{11})d_k(P_2)\} \\
&= \sum_{i+j+k=n,1\le i,j,k} \{d_i(X_{11})d_j(X_{22})d_k(P_2) + d_i(X_{22})d_j(X_{11})d_k(P_2)\} \\
&\quad + \sum_{j+k=n,1\le j,k} \{X_{11}d_j(X_{22})d_k(P_2) + X_{22}d_j(X_{11})d_k(P_2)\} \\
&\quad + \sum_{i+k=n,1\le i,k} \{d_i(X_{11})X_{22}d_k(P_2) + d_i(X_{22})X_{11}d_k(P_2)\} \\
&\quad + \sum_{i+j=n,1\le i,j} \{d_i(X_{11})d_j(X_{22})P_2 + d_i(X_{22})d_j(X_{11})P_2\} \\
&\quad + d_n(X_{11})X_{22} + X_{11}d_n(X_{22})P_2 \\
&= \sum_{i+j=n,1\le i,j} d_i(X_{11})d_j(X_{22}) + d_n(X_{11})X_{22} + X_{11}d_n(X_{22}) \\
&= \sum_{i+j=n} d_i(X_{11})d_j(X_{22}).
\end{aligned}$$

(ii) For any $n \in \mathbb{N}^+$, $X_{11} \in \mathcal{A}$, $X_{12} \in \mathcal{M}$, since $X_{11}X_{12}P_2 = X_{11}X_{12} \in \Omega$, we take $X = X_{11}, Y = X_{12}, Z = P_2$ in Eq. (1). Then we can get from the properties (P2), Lemmas 2 and 3 that

$$\begin{aligned}
d_n(X_{11}X_{12}) &= d_n(X_{11}X_{12}P_2 + X_{12}X_{11}P_2) \\
&= \sum_{i+j+k=n} \{d_i(X_{11})d_j(X_{12})d_k(P_2) + d_i(X_{12})d_j(X_{11})d_k(P_2)\} \\
&= \sum_{i+j+k=n,1\le i,j,k} \{d_i(X_{11})d_j(X_{12})d_k(P_2) + d_i(X_{12})d_j(X_{11})d_k(P_2)\} \\
&\quad + \sum_{j+k=n,1\le j,k} \{X_{11}d_j(X_{12})d_k(P_2) + X_{12}d_j(X_{11})d_k(P_2)\} \\
&\quad + \sum_{i+k=n,1\le i,k} \{d_i(X_{11})X_{12}d_k(P_2) + d_i(X_{12})X_{11}d_k(P_2)\} \\
&\quad + \sum_{i+j=n,1\le i,j} \{d_i(X_{11})d_j(X_{12})P_2 + d_i(X_{12})d_j(X_{11})P_2\} \\
&\quad + d_n(X_{11})X_{12} + X_{11}d_n(X_{12})P_2 \\
&= \sum_{i+j=n,1\le i,j} d_i(X_{11})d_j(X_{12}) + d_n(X_{11})X_{12} + X_{11}d_n(X_{12}) \\
&= \sum_{i+j=n} d_i(X_{11})d_j(X_{12}).
\end{aligned}$$

Similarly, we can show (iii) holds.

(iv) For any $n \in \mathbb{N}^+$, $X_{11}, Y_{11} \in \mathcal{A}, Z_{12} \in \mathcal{M}$, it follows from Lemma 4 (ii) and the properties (P2) that

$$\begin{aligned} d_n(X_{11}Y_{11}Z_{12}) &= d_n((X_{11}Y_{11})Z_{12}) = \sum_{i+j=n} d_i(X_{11}Y_{11})d_j(Z_{12}) \\ &= \sum_{i+j=n,1\leq j} d_i(X_{11}Y_{11})d_j(Z_{12}) + d_n(X_{11}Y_{11})Z_{12} \\ &= \sum_{i+j=n,1\leq j} (\sum_{p+q=i} d_p(X_{11})d_q(Y_{11}))d_j(Z_{12}) + d_n(X_{11}Y_{11})Z_{12} \\ &= \sum_{p+q+j=n,1\leq j} d_p(X_{11})d_q(Y_{11})d_j(Z_{12}) + d_n(X_{11}Y_{11})Z_{12}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & d_n(X_{11}Y_{11}Z_{12}) \\ &= d_n(X_{11}(Y_{11}Z_{12})) = \sum_{i+j=n} d_i(X_{11})d_j(Y_{11}Z_{12}) \\ &= \sum_{i+j=n} d_i(X_{11}) \sum_{p+q=j} d_p(Y_{11})d_q(Z_{12}) \\ &= \sum_{p+q+j=n} d_p(X_{11})d_q(Y_{11})d_j(Z_{12}) \\ &= \sum_{p+q+j=n,1\leq j} d_p(X_{11})d_q(Y_{11})d_j(Z_{12}) + \sum_{p+q=n} d_p(X_{11})d_q(Y_{11})Z_{12}. \end{aligned}$$

Comparing above two equations, and by the faithfulness of $\mathcal{M}$, we get that

$$P_1 d_n(X_{11}Y_{11})P_1 = P_1 \sum_{i+j=n} d_i(X_{11})d_j(Y_{11})P_1. \tag{6}$$

In the following, we will show $P_1 d_n(X_{11}Y_{11})P_2 = P_1 \sum_{i+j=n} d_i(X_{11}) d_j(Y_{11})P_2$ holds. Indeed, For any $n \in \mathbb{N}^+$, $X_{11}, Y_{11} \in \mathcal{A}$, by Lemma 4(i) and the properties (P2), we get

$$\begin{aligned} 0 &= d_n((X_{11}Y_{11})P_2) \\ &= \sum_{i+j=n} d_i(X_{11}Y_{11})d_j(P_2) \\ &= \sum_{i+j=n,1\leq j} d_i(X_{11}Y_{11})d_j(P_2) + d_n(X_{11}Y_{11})P_2 \\ &= \sum_{i+j=n,1\leq j} \{\sum_{p+q=i} d_p(X_{11})d_q(Y_{11})\}d_j(P_2) + d_n(X_{11}Y_{11})P_2 \\ &= \sum_{p+q+j=n,1\leq j} d_p(X_{11})d_q(Y_{11})d_j(P_2) + d_n(X_{11}Y_{11})P_2 \end{aligned}$$

$$= \sum_{p=0}^{n} d_p(X_{11})\{\sum_{q+j=n-p} d_q(Y_{11})d_j(P_2)\}$$
$$- \sum_{p+q=n} d_p(X_{11})d_q(Y_{11})P_2 + d_n(X_{11}Y_{11})P_2$$
$$= \sum_{p=0}^{n} d_p(X_{11})d_{n-p}(Y_{11}P_2) - \sum_{p+q=n} d_p(X_{11})d_q(Y_{11})P_2 + d_n(X_{11}Y_{11})P_2$$
$$= - \sum_{p+q=n} d_p(X_{11})d_q(Y_{11})P_2 + d_n(X_{11}Y_{11})P_2.$$

This implies that

$$P_1 d_n(X_{11}Y_{11})P_2 = P_1 \sum_{i+j=n} d_i(X_{11})d_j(Y_{11})P_2. \tag{7}$$

Therefore, by Eqs. (6) and (7) and Lemma 3, we get $d_n(X_{11}Y_{11}) = \sum_{i+j=n} d_i(X_{11})d_j(Y_{11})$. Similarly, we can show (v) holds. The proof is completed. ■

Lemma 5. *For any $n \in \mathbb{N}^+$, $X_{11}, Y_{11} \in \mathcal{A}$, $X_{12}, Y_{12} \in \mathcal{M}$ and $X_{22}, Y_{22} \in \mathcal{B}$, the following statements hold:*

(i) $d_n(X_{11} + X_{12}) = d_n(X_{11}) + d_n(X_{12})$;
(ii) $d_n(Y_{12} + Y_{22}) = d_n(Y_{12}) + d_n(Y_{22})$;
(iii) $d_n(X_{12} + Y_{12}) = d_n(X_{12}) + d_n(Y_{12})$;
(iv) $d_n(X_{11} + Y_{11}) = d_n(X_{11}) + d_n(Y_{11})$;
(v) $d_n(X_{22} + Y_{22}) = d_n(X_{22}) + d_n(Y_{22})$.

Proof. (i) For any $n \in \mathbb{N}^+$, $X_{11} \in \mathcal{A}$ and $X_{12} \in \mathcal{M}$, since $P_1(X_{11} + X_{12})P_2 = X_{12} \in \Omega$, we take $X = P_1, Y = X_{11} + X_{12}, Z = P_2$ in Eq. (1). Then by Lemmas 2–4, and the properties (P2), we get

$$d_n(X_{12})$$
$$= d_n(P_1(X_{11} + X_{12})P_2 + (X_{11} + X_{12})P_1P_2)$$
$$= \sum_{i+j+k=n} \{d_i(P_1)d_j(X_{11} + X_{12})d_k(P_2) + d_i(X_{11} + X_{12})d_j(P_1)d_k(P_2)\}$$
$$= \sum_{i+j+k=n,1\leq i,j,k} \{d_i(P_1)d_j(X_{11} + X_{12})d_k(P_2) + d_i(X_{11} + X_{12})d_j(P_1)d_k(P_2)\}$$
$$+ \sum_{j+k=n,1\leq j,k} \{P_1 d_j(X_{11} + X_{12})d_k(P_2) + (X_{11} + X_{12})d_j(P_1)d_k(P_2)\}$$
$$+ \sum_{i+k=n,1\leq i,k} \{d_i(P_1)(X_{11} + X_{12})d_k(P_2) + d_i(X_{11} + X_{12})P_1 d_k(P_2)\}$$
$$+ \sum_{i+j=n,1\leq i,j} \{d_i(P_1)d_j(X_{11} + X_{12})P_2 + d_i(X_{11} + X_{12})d_j(P_1)P_2\}$$

$$
\begin{aligned}
&+P_1 d_n(X_{11}+X_{12})P_2 + X_{11}d_n(P_1)P_2 + 2X_{11}d_n(P_2)\\
=& \sum_{j+k=n,1\leq j,k} d_j(X_{11})d_k(P_2) + \sum_{i+k=n,1\leq i,k} d_i(X_{11})d_k(P_2)\\
&+ \sum_{i+j=n,1\leq i,j} d_i(X_{11})d_j(P_1)\\
&+P_1 d_n(X_{11}+X_{12})P_2 + X_{11}d_n(P_1) + 2X_{11}d_n(P_2)\\
=& \sum_{j+k=n} d_j(X_{11})d_k(P_2) + \sum_{i+k=n} d_i(X_{11})d_k(P_2) + \sum_{i+j=n} d_i(X_{11})d_j(P_1)\\
&+P_1 d_n(X_{11}+X_{12})P_2 - d_n(X_{11})P_1 - 2d_n(X_{11})P_2\\
=& \, d_n(X_{11}) + P_1 d_n(X_{11}+X_{12})P_2 - d_n(X_{11})P_1 - 2d_n(X_{11})P_2\\
=& \, P_1 d_n(X_{11}+X_{12})P_2 - d_n(X_{11})P_2
\end{aligned}
$$

This implies that

$$P_1 d_n(X_{11}+X_{12})P_2 = P_1 d_n(X_{11})P_2 + d_n(X_{12}). \tag{8}$$

Similarly, we can get that

$$P_2 d_n(X_{11}+X_{12})P_2 = 0. \tag{9}$$

In the following, we will show $P_1 d_n(X_{11}+X_{12})P_1 = P_1 d_n(X_{11})P_1$ holds. For any $n \in \mathbb{N}^+$, $X_{12}, Z_{12} \in \mathcal{M}$, since $(X_{11}+X_{12})Z_{12}P_2 = X_{11}Z_{12} \in \Omega$, we take $X = X_{11}+X_{12}, Y = Z_{12}, Z = P_2$ in Eq. (1). Then by Lemmas 2–3, Lemma 4(ii), $P_2 d_n(X_{11}+X_{12})P_2 = 0$ and the properties (P2), we get

$$
\begin{aligned}
& d_n(X_{11}Z_{12})\\
=& \, d_n((X_{11}+X_{12})Z_{12}P_2 + Z_{12}(X_{11}+X_{12})P_2)\\
=& \sum_{i+j+k=n} \{d_i(X_{11}+X_{12})d_j(Z_{12})d_k(P_2)\\
&+d_i(Z_{12})d_j(X_{11}+X_{12})d_k(P_2)\}\\
=& \sum_{i+j+k=n,1\leq i,j,k} \{d_i(X_{11}+X_{12})d_j(Z_{12})d_k(P_2)\\
&+d_i(Z_{12})d_j(X_{11}+X_{12})d_k(P_2)\}\\
&+ \sum_{j+k=n,1\leq j,k} \{(X_{11}+X_{12})d_j(Z_{12})d_k(P_2) + Z_{12}d_j(X_{11}+X_{12})d_k(P_2)\}\\
&+ \sum_{i+k=n,1\leq i,k} \{d_i(X_{11}+X_{12})Z_{12}d_k(P_2) + d_i(Z_{12})(X_{11}+X_{12})d_k(P_2)\}\\
&+ \sum_{i+j=n,1\leq i,j} \{d_i(X_{11}+X_{12})d_j(Z_{12})P_2 + d_i(Z_{12})d_j(X_{11}+X_{12})P_2\}\\
&+d_n(X_{11}+X_{12})Z_{12} + X_{11}d_n(Z_{12})\\
=& \sum_{i+j=n,1\leq i,j} d_i(X_{11})d_j(Z_{12}) + d_n(X_{11}+X_{12})Z_{12} + X_{11}d_n(Z_{12})
\end{aligned}
$$

$$= \sum_{i+j=n} d_i(X_{11})d_j(Z_{12}) + d_n(X_{11}+X_{12})Z_{12} - d_n(X_{11})Z_{12}$$
$$= d_n(X_{11}Z_{12}) + d_n(X_{11}+X_{12})Z_{12} - d_n(X_{11})Z_{12}.$$

And then by the faithfulness of $\mathcal{M}$, we get that

$$P_1 d_n(X_{11}+X_{12})P_1 = P_1 d_n(X_{11})P_1. \tag{10}$$

Therefore, by Eqs. (8)-(10) and Lemmas 2-3, we have $d_n(X_{11}+X_{12}) = d_n(X_{11}) + d_n(X_{12})$. Similarly, we can show (ii) holds.

(iii) For any $n \in \mathbb{N}^+$, $X_{12}, Y_{12} \in \mathcal{M}$, since $(P_1+X_{12})(Y_{12}+P_2)P_2 = X_{12} + Y_{12} \in \Omega$, we take $X = (P_1+X_{12}), Y = (Y_{12}+P_2), Z = P_2$ in Eq. (1). Then by Lemmas 2-4, Lemma 5(i)-(ii), and the properties (P2), we get

$$\begin{aligned}
d_n(X_{12}+Y_{12}) &= d_n((P_1+X_{12})(Y_{12}+P_2)P_2 + (Y_{12}+P_2)(P_1+X_{12})P_2)\\
&= \sum_{i+j+k=n} \{d_i(P_1+X_{12})d_j(Y_{12}+P_2)d_k(P_2)\\
&\quad + d_i(Y_{12}+P_2)d_j(P_1+X_{12})d_k(P_2)\}\\
&= \sum_{i+j+k=n,1\le i,j,k} \{d_i(P_1+X_{12})d_j(Y_{12}+P_2)d_k(P_2)\\
&\quad + d_i(Y_{12}+P_2)d_j(P_1+X_{12})d_k(P_2)\}\\
&\quad + \sum_{j+k=n,1\le j,k} \{(P_1+X_{12})d_j(Y_{12}+P_2)d_k(P_2)\\
&\quad + (Y_{12}+P_2)d_j(P_1+X_{12})d_k(P_2)\}\\
&\quad + \sum_{i+k=n,1\le i,k} \{d_i(P_1+X_{12})(Y_{12}+P_2)d_k(P_2)\\
&\quad + d_i(Y_{12}+P_2)(P_1+X_{12})d_k(P_2)\}\\
&\quad + \sum_{i+j=n,1\le i,j} \{d_i(P_1+X_{12})d_j(Y_{12}+P_2)P_2\\
&\quad + d_i(Y_{12}+P_2)d_j(P_1+X_{12})P_2\}\\
&\quad + d_n(X_{12}) + d_n(P_1) + d_n(Y_{12}) + d_n(P_2)\\
&= d_n(X_{12}) + d_n(P_1) + d_n(Y_{12}) + d_n(P_2)\\
&= d_n(X_{12}) + d_n(Y_{12}).
\end{aligned}$$

(iv) For any $n \in \mathbb{N}^+$, $X_{11}, Y_{11} \in \mathcal{A}, Z_{12} \in \mathcal{M}$, then by Lemmas 2-5(iii), and the properties (P2), we get

$$\begin{aligned}
& d_n(X_{11}Z_{12}) + d_n(Y_{11}Z_{12})\\
&= d_n((X_{11}+Y_{11})Z_{12})\\
&= \sum_{i+j=n} d_i(X_{11}+Y_{11})d_j(Z_{12})\\
&= \sum_{i+j=n,1\le i,j} (d_i(X_{11}) + d_i(Y_{11}))d_j(Z_{12})
\end{aligned}$$

$$
\begin{aligned}
&+d_n(X_{11}+Y_{11})Z_{12}+(X_{11}+Y_{11})d_n(Z_{12})\\
=&\sum_{i+j=n} d_i(X_{11})d_j(Z_{12})-d_n(X_{11})Z_{12}\\
&+\sum_{i+j=n} d_i(Y_{11})d_j(Z_{12})-d_n(Y_{11})Z_{12}+d_n(X_{11}+Y_{11})Z_{12}\\
=&\ d_n(X_{11}Z_{12})+d_n(Y_{11}Z_{12})\\
&+(d_n(X_{11}+Y_{11})-d_n(X_{11})-d_n(Y_{11}))Z_{12}.
\end{aligned}
$$

Therefore, by the faithfulness of $\mathcal{M}$, we get that

$$
P_1d_n(X_{11}+Y_{11})P_1 = P_1d_n(X_{11})P_1 + P_1d_n(Y_{11})P_1. \tag{11}
$$

On the other hand, it follows from Lemma 4(i) and the properties (P2) that

$$
\begin{aligned}
0 =&\ d_n(X_{11}+Y_{11})P_2)\\
=&\sum_{i+j=n} d_i(X_{11}+Y_{11})d_j(P_2)\\
=&\sum_{i+j=n,1\leq i,j} \{d_i(X_{11})+d_i(Y_{11})\}d_j(P_2)\\
&+d_n(X_{11}+Y_{11})P_2+(X_{11}+Y_{11})d_n(P_2)\\
=&\sum_{i+j=n} d_i(X_{11})d_j(P_2)+\sum_{i+j=n} d_i(Y_{11})d_j(P_2)\\
&+d_n(X_{11}+Y_{11})P_2-d_n(X_{11})P_2-d_n(Y_{11})P_2\\
=&\ d_n(X_{11}+Y_{11})P_2-d_n(X_{11})P_2-d_n(Y_{11})P_2.
\end{aligned}
$$

This implies that

$$
P_1d_n(X_{11}+Y_{11})P_2 = P_1d_n(X_{11})P_2 + P_1d_n(Y_{11})P_2. \tag{12}
$$

Therefore, by Eqs. (11)-(12) and Lemma 3, we obtain $d_n(X_{11}+Y_{11}) = d_n(X_{11})+d_n(Y_{11})$. Similarly, we can show (v) holds. The proof is completed. ■

Lemma 6. *For any $n \in \mathbb{N}^+$,$X_{11} \in \mathcal{A}, X_{12} \in \mathcal{M}$ and $X_{22} \in \mathcal{B}$, we have*

$$
d_n(X_{11}+X_{12}+X_{22}) = d_n(X_{11})+d_n(X_{12})+d_n(X_{22}).
$$

Proof. For any $n \in \mathbb{N}^+$,$X_{11} \in \mathcal{A}, X_{12} \in \mathcal{M}$ and $X_{22} \in \mathcal{B}$, since $P_1(X_{11}+X_{12}+X_{22})P_2 = X_{12} \in \Omega$, we take $X = P_1, Y = X_{11}+X_{12}+X_{22}, Z = P_2$ in Eq. (1).

Then by Lemmas 2-4, and the properties (P2), we get

$$\begin{aligned}
& d_n(X_{12}) \\
=\ & d_n(P_1(X_{11}+X_{12}+X_{22})P_2+(X_{11}+X_{12}+X_{22})P_1P_2) \\
=\ & \sum_{i+j+k=n}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})d_k(P_2) \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(P_2)\} \\
=\ & \sum_{i+j+k=n,1\leq i,j,k}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})d_k(P_2) \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(P_2)\} \\
& +\sum_{j+k=n,1\leq j,k}\{P_1d_j(X_{11}+X_{12}+X_{22})d_k(P_2) \\
& +(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(P_2)\} \\
& +\sum_{i+k=n,1\leq i,k}\{d_i(P_1)(X_{11}+X_{12}+X_{22})d_k(P_2) \\
& +d_i(X_{11}+X_{12}+X_{22})P_1d_k(P_2)\} \\
& +\sum_{i+j=n,1\leq i,j}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})P_2 \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)P_2\} \\
& +d_n(P_1)X_{22}+P_1d_n(X_{11}+X_{12}+X_{22})P_2+X_{11}d_n(P_1)P_2+2X_{11}d_n(P_2) \\
=\ & \sum_{j+k=n,1\leq j,k}d_j(X_{11})d_k(P_2)+\sum_{i+k=n,1\leq i,k}d_i(X_{11})d_k(P_2) \\
& +\sum_{i+j=n,1\leq i,j}d_i(P_1)d_j(X_{22})+\sum_{i+j=n,1\leq i,j}d_i(X_{11})d_j(P_1) \\
& +d_n(P_1)X_{22}+P_1d_n(X_{11}+X_{12}+X_{22})P_2+X_{11}d_n(P_1)+2X_{11}d_n(P_2) \\
=\ & 2\sum_{j+k=n}d_j(X_{11})d_k(P_2)+\sum_{i+j=n}d_i(P_1)d_j(X_{22})+\sum_{i+j=n}d_i(X_{11})d_j(P_1) \\
& +P_1d_n(X_{11}+X_{12}+X_{22})P_2-2d_n(X_{11})P_2-P_1d_n(X_{22})-d_n(X_{11})P_1 \\
=\ & d_n(X_{11})+P_1d_n(X_{11}+X_{12}+X_{22})P_2 \\
& -2d_n(X_{11})P_2-P_1d_n(X_{22})-d_n(X_{11})P_1 \\
=\ & P_1d_n(X_{11}+X_{12}+X_{22})P_2-d_n(X_{11})P_2-P_1d_n(X_{22}).
\end{aligned}$$

This implies that

$$P_1d_n(X_{11}+X_{12}+X_{22})P_2=P_1d_n(X_{11})P_2+d_n(X_{12})+P_1d_n(X_{22})P_2. \quad (13)$$

In the following, we will show $P_1d_n(X_{11}+X_{12}+X_{22})P_1 = P_1d_n(X_{11})P_1$ and $P_2d_n(X_{11}+X_{12}+X_{22})P_2 = P_2d_n(X_{22})P_2$ hold. Indeed, For any $n \in \mathbb{N}^+$, $Z_{12} \in \mathcal{M}$, since $P_1(X_{11}+X_{12}+X_{22})Z_{12} = X_{11}Z_{12} \in \Omega$, we take $X = P_1, Y = X_{11}+X_{12}+X_{22}, Z = Z_{12}$ in Eq. (1). Then by Lemmas 2-5, and the properties

(P2), we get

$$
\begin{aligned}
& d_n(2X_{11}Z_{12}) \\
= & d_n(P_1(X_{11}+X_{12}+X_{22})Z_{12}+(X_{11}+X_{12}+X_{22})P_1Z_{12}) \\
= & \sum_{i+j+k=n}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})d_k(Z_{12}) \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(Z_{12})\} \\
= & \sum_{i+j+k=n,1\leq i,j,k}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})d_k(Z_{12}) \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(Z_{12})\} \\
& +\sum_{j+k=n,1\leq j,k}\{P_1d_j(X_{11}+X_{12}+X_{22})d_k(Z_{12}) \\
& +(X_{11}+X_{12}+X_{22})d_j(P_1)d_k(Z_{12})\} \\
& +\sum_{i+k=n,1\leq i,k}\{d_i(P_1)(X_{11}+X_{12}+X_{22})d_k(Z_{12}) \\
& +d_i(X_{11}+X_{12}+X_{22})P_1d_k(Z_{12})\} \\
& +\sum_{i+j=n,1\leq i,j}\{d_i(P_1)d_j(X_{11}+X_{12}+X_{22})Z_{12} \\
& +d_i(X_{11}+X_{12}+X_{22})d_j(P_1)Z_{12}\} \\
& +2P_1d_n(X_{11}+X_{12}+X_{22})Z_{12}+2X_{11}d_n(Z_{12}) \\
= & \sum_{j+k=n,1\leq j,k}d_j(X_{11})d_k(Z_{12}) \\
& +\sum_{i+k=n,1\leq i,k}d_i(X_{11})P_1d_k(Z_{12}) \\
& +2P_1d_n(X_{11}+X_{12}+X_{22})Z_{12}+2X_{11}d_n(Z_{12}) \\
= & 2\sum_{i+j=n}d_i(X_{11})d_j(Z_{12})+2P_1d_n(X_{11}+X_{12}+X_{22})Z_{12}-2d_n(X_{11})Z_{12} \\
= & d_n(2X_{11}Z_{12})+2P_1(d_n(X_{11}+X_{12}+X_{22})-d_n(X_{11}))Z_{12}.
\end{aligned}
$$

Hence, by the property of 2-torsion free of $\mathcal{U}$ and the faithfulness of $\mathcal{M}$, we get that

$$P_1d_n(X_{11}+X_{12}+X_{22})P_1=P_1d_n(X_{11})P_1. \tag{14}$$

Similarly, we can get that

$$P_2d_n(X_{11}+X_{12}+X_{22})P_2=P_2d_n(X_{22})P_2. \tag{15}$$

Therefore, by Eqs. (13)-(15), we get $d_n(X_{11}+X_{12}+X_{22})=d_n(X_{11})+d_n(X_{12})+d_n(X_{22})$. ■

Now, we begin to prove Theorem 1.

Proof of Theorem 1. It follows from Lemmas 5 and 6 that $D = \{d_n\}_{n\in\mathbb{N}}$ has the additivity on $\mathcal{U}$. Now, we show that $D = \{d_n\}_{n\in\mathbb{N}}$ is an additive higher derivation on $\mathcal{U}$. Let $X = X_{11} + X_{12} + X_{22}$ and $Y = Y_{11} + Y_{12} + Y_{22}$ be arbitrary elements of $\mathcal{U}$, where $X_{11}, Y_{11} \in \mathcal{A}, X_{12}, Y_{12} \in \mathcal{M}$ and $X_{22}, Y_{22} \in \mathcal{B}$. Since $D = \{d_n\}_{n\in\mathbb{N}}$ has the additivity on $\mathcal{U}$, so we can obtain from Lemmas 2-4 that

$$\begin{aligned} d_n(XY) &= d_n((X_{11} + X_{12} + X_{22})(Y_{11} + Y_{12} + Y_{22})) \\ &= d_n(X_{11}Y_{11} + X_{11}Y_{12} + X_{12}Y_{22} + X_{22}Y_{22}) \\ &= d_n(X_{11}Y_{11}) + d_n(X_{11}Y_{12}) + d_n(X_{12}Y_{22}) + d_n(X_{22}Y_{22}) \\ &= \sum_{i+j=n} d_i(X_{11})d_j(Y_{11}) + \sum_{i+j=n} d_i(X_{11})d_j(Y_{12}) + \sum_{i+j=n} d_i(X_{11})d_j(Y_{22}) \\ &\quad + \sum_{i+j=n} d_i(X_{12})d_j(Y_{22}) + \sum_{i+j=n} d_i(X_{22})d_j(Y_{22}) \\ &= \sum_{i+j=n} d_i(X)d_j(Y) \end{aligned}$$

Therefore, $D = \{d_n\}_{n\in\mathbb{N}}$ is an additive higher derivation on $\mathcal{U}$. The proof is completed. ■

Next we give an application of Theorem 1 to certain special classes of triangular algebras, such as block upper triangular matrix algebras and nest algebras.

Let $\mathcal{R}$ be a commutative ring with identity and let $M_{n\times k}(\mathcal{R})$ be the set of all $n \times k$ matrices over $\mathcal{R}$. For $n \geq 2$ and $m \leq n$, the block upper triangular matrix algebra $T_n^{\bar{k}}(\mathcal{R})$ is a subalgebra of $M_n(\mathcal{R})$ with the form

$$\begin{pmatrix} M_{k_1}(\mathcal{R}) & M_{k_1\times k_2}(\mathcal{R}) & \cdots & M_{k_1\times k_m}(\mathcal{R}) \\ 0 & M_{k_2}(\mathcal{R}) & \cdots & M_{k_2\times k_m}(\mathcal{R}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathcal{R}) \end{pmatrix},$$

where $\bar{k} = (k_1, k_2, \cdots, k_m)$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

A nest of a complex Hilbert space H is a chain $\mathcal{N}$ of closed subspaces of H containing $\{0\}$ and H which is closed under arbitrary intersections and closed linear span. The nest algebra associated to $\mathcal{N}$ is the algebra

$$\text{Alg}\mathcal{N} = \{T \in B(H) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, H\}$. It is clear that every nontrivial nest algebra is a triangular algebra and every finite dimensional nest algebra is isomorphic to a complex block upper triangular matrix algebra.

Corollary 7. *Let $T_n^{\bar{k}}(\mathcal{R})$ be a 2-torsion free block upper triangular matrix algebra, $D = \{d_n\}_{n\in\mathbb{N}}$ be a nonlinear non-global semi-Jordan triple higher derivable mapping on $T_n^{\bar{k}}(\mathcal{R})$. Then $D = \{d_n\}_{n\in\mathbb{N}}$ is an additive higher derivation.*

Corollary 8. *Let $\mathcal{N}$ be a nontrivial nest of a complex Hilbert space H and* Alg$\mathcal{N}$ *be a nest algebra, $D = \{d_n\}_{n\in\mathbb{N}}$ be a nonlinear non-global semi-Jordan triple higher derivable mapping on* Alg$\mathcal{N}$. *Then $D = \{d_n\}_{n\in\mathbb{N}}$ is an additive higher derivation.*

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