Southeast Asian Bulletin of Mathematics (2024) 48: 211–226

Southeast Asian
Bulletin of
Mathematics

Galois Theory for Soft Int-field

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Received 16 May 2021
Accepted 15 December 2021

Communicated by L.A. Bokut

AMS Mathematics Subject Classification(2020): 06A15, 12F10, 06D72

Abstract. In this paper, the notion of soft normal int-field over a field has been introduced. We have established a correspondence, called Soft Galois correspondence, between the soft intermediate int-fields of a finite Galois extension and the soft int-groups of the Galois group corresponding to the field extension. We have generalized some results of Galois theory in the environment of soft set theory.

Keywords: Soft set; Soft int-field; Soft normal int-field; Soft Galois correspondence.

1. Introduction

Soft set theory was initiated by Molodtsov [21] in 1999 as a parameterized mathematical tool for modeling uncertainties. Theoretical development of soft set theory and its applications in decision making problems [19, 22] both have been done tremendously during last two decades.

Algebraic structure in soft set theory was first introduced by Aktas et al. [2] in 2007 and they defined soft group as a parameterized family of subgroups.

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With the introduction of soft group, a new direction was revealed for researchers. Hence the notions of soft semiring [9], soft ring [1], soft field [18], soft vector space [23], soft module [25] etc. have been developed rapidly. Also, the notions of fuzzy soft group [4], fuzzy soft ring [8, 10] etc. were studied simultaneously.

But many results of classical group theory, ring theory etc. could not be verified properly in soft set theory using the above approach. That means development of different algebraic structures from classical set theory to soft set theory in this approach is not up to the mark. In 2012 Cagman et al. [6] developed the notion of group structure in soft set theory using inclusion operators and hence the notion of soft int-group was established. Thereafter the notions of soft int-ring [3, 7, 14], soft int-field [3], soft int-ideal [11, 12, 13] etc. have been established. Although the study of field structure in soft set theory is not sufficient. Recently, Ghosh et al. [15] has studied different properties of soft int-field extension.

In this paper, section 3 introduces the notion of soft normal int-field over a field and illustrates it with suitable example. In section 4, we define a Soft Galois correspondence between the soft intermediate int-fields of a finite Galois extension and the soft int-groups of the associated Galois group. Hence we generalize some results of Galois theory in soft set setting using Soft Galois correspondence.

2. Preliminaries

Some basic definitions and results of classical set theory and soft set theory are collected here for use in the later sections. Throughout this paper, unless otherwise stated, let U refer to an initial universe, E the set of parameters, $P(U)$ the power set of $U, \mathbb{N}$ the set of all natural numbers and $A \subseteq E$.

Definition 2.1. [21] *A pair (F, A) is called a soft set of A over U, where F is a mapping given by $F : A \to P(U)$. The soft set (F, A) is simply denoted by F, when no confusions regarding the parameter set A and the universal set U arise. The collection of all soft sets with parameter set A over U will be denoted by $S(A, U)$.*

Definition 2.2. [6] *Let $F \in S(A, U)$ and $K \subseteq U$. Then the set $F_K = \{x \in A : F(x) \supseteq K\}$ is called K-inclusion subset of the soft set F. Here, we denote the set $\{x \in A : F(x) \supset K\}$ by $F_{K^\star}$.*

Definition 2.3. *Let $F \in S(A, U)$. The image of F is denoted by $Im(F)$ and defined by $Im(F) = \{F(x) : x \in A\}$.*

Definition 2.4. [24] *Let $A \subseteq E$. The soft characteristic function of A over U,* *denoted by* χ_A, *is defined by the soft set* $\chi_A : E \to P(U)$, *where*

$$\chi_A(x) = \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Definition 2.5. [6] *Let* $F \in S(A,U)$ *and* $f : A \to A^{'}$ *be any mapping, where* $A, A^{'}$ *are parameter sets. Then the image of* F *under* f *is denoted by* $f(F)$ *and defined for all* $y \in A^{'}$ *by*

$$f(F)(y) = \begin{cases} \bigcup\limits_{x \in f^{-1}(y)} F(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorem 2.6. [6] *Let* G *be any group. A soft set* $H \in S(G,U)$ *is a soft int-group of* G *if and only if* $H(xy^{-1}) \supseteq H(x) \cap H(y)$ *for all* $x, y \in G$.

Theorem 2.7. [6] *Let* G *be a group. A soft set* $H \in S(G,U)$ *is a soft int-group of* G *if and only if* H_K *is a subgroup of* G *for all* $K \subseteq H(e)$, *where* e *is the identity element of* G.

Theorem 2.8. [6] *Let* G *be a group and* $H \in S(G,U)$. *Then* H *is a soft normal int-group of* G *if and only if* H_L *is a normal subgroup of* G *for all* $L \subseteq H(e)$, *where* e *is the identity element of* G.

Definition 2.9. [6] *Let* H *be a soft int-group of a group* G *over* U *and* $a \in G$. *Then the soft set* H_a *of* G, *defined by* $H_a(x) = H(xa^{-1})$ *for all* $x \in G$, *is called the soft coset of* H *in* G *determined by* $a \in G$.

Theorem 2.10. [6] *Let us denote the set* $\{H_a : a \in G\}$ *by* G/H. *If* H *is a soft normal int-group of* G *then* G/H *forms a group with respect to the binary composition* $H_a * H_b = H_{ab}$ *for all* $a, b \in G$. *Moreover,* G/H *is isomorphic to* G/H_L, *where* $L = H(e)$ *and* e *is the identity element of* G.

Here $|G/H|$ is called the index of H in G and it is denoted by $[G : H]$.

Definition 2.11. [3] *Let* T *be a field. A soft set* $F \in S(T,U)$ *is called a soft int-field of* T *if* $F(x-y) \supseteq F(x) \cap F(y)$ *and* $F(xy^{-1}) \supseteq F(x) \cap F(y)$, *for all* $x, y \in T$, *where* $y \neq 0$ *(the zero element of* T*).*

Proposition 2.12. *Let* T *be a field and* F *a soft int-field of* T. *Then* $F(0) \supseteq F(1) \supseteq F(r) = F(-r) = F(r^{-1})$ *for all* $r (\neq 0) \in T$, *where* 0 *is the zero element of* T.

Theorem 2.13. [15] *A soft set F of a field T is a soft int-field of T if and only if K-inclusion subsets F_K are subfields of T for all $K \subseteq F(0)$.*

Theorem 2.14. [16] *Let A/B be a field extension and $G(A/B)$ be the collection of all $B-$automorphism of A, i.e., $G(A/B) = \{\alpha : A \to A \text{is an isomorphism} : \alpha(b) = b, \text{ for all } b \in B\}$. Then $G(A/B)$ forms a group w.r.t. the composition of mappings.*

The group $G(A/B)$ is called the Galois group of A over B.

Theorem 2.15. [5, 16] *Let A/B be a normal field extension. An intermediate field C of A/B is normal over B if and only if $\alpha(C) = C$ for all $\alpha \in G(A/B)$.*

Proposition 2.16. [16] *Let A/B be a field extension. If P is a subgroup of $G(A/B)$ then the set $A^P = \{a \in A : \alpha(a) = a \text{ for all } \alpha \in P\}$ is an intermediate field of A/B. The field A^P is called the fixed field of the subgroup P.*

3. Soft Normal Int-field over a Field

Form this section, unless otherwise stated, all soft sets are to be considered over U. We shall fix some notations which will be used thereafter.

Definition 3.1. *Let A/B be a field extension. A soft int-field F of A is called soft intermediate int-field of A/B if $F_K \supseteq B$, for all $K \in Im(F)$.*

Note 3.2. For a field extension $A/B, (A/B)^{\clubsuit}$ denotes the collection of all soft intermediate int-fields of A/B and $G(A/B)^{\star}$ denotes the collection of all soft int-groups of $G(A/B)$.

Let $\alpha : A \to A$ be an isomorphism and F be a soft int-field of A. Then for each $a \in A, \alpha^{-1}(a)$ contain exactly one element, say $a^{'} \in A$. Then from Definition 2.5, we have

$$\alpha(F)(a) = \bigcup_{x \in \alpha^{-1}(a)} F(x) = F(a^{'}), \text{ for all } \ a \in A. \tag{1}$$

Proposition 3.3. *Let $\alpha : A \to A$ be an isomorphism and F be a soft int-field of A. Then $\alpha(F)$ is a soft int-field of A.*

Proof. From Eq. (1), it is clear that $\alpha(F)$ is a soft subset of A. Let $a, b \in A$. Then $\alpha(F)(a) = F(a^{'})$ where $a^{'} \in A$ such that $\alpha(a^{'}) = a$. Similarly, $\alpha(F)(b) = F(b^{'})$ where $b^{'} \in A$ such that $\alpha(b^{'}) = b$. Also, $\alpha(F)(a - b) = F(r^{'})$ where $r^{'} \in A$

such that $\alpha(r') = a - b$. Since α is an isomorphism, we have $\alpha(r') = a - b = \alpha(a') - \alpha(b') = \alpha(a' - b')$ and hence $r' = a' - b'$. Therefore $\alpha(F)(a - b) = F(r') = F(a' - b') \supseteq F(a') \cap F(b') \supseteq \alpha(F)(a) \cap \alpha(F)(b)$. Similarly, we can prove that $\alpha(F)(cd^{-1}) \supseteq \alpha(F)(c) \cap \alpha(F)(d)$ for $c, d (\neq 0) \in A$, Therefore $\alpha(F)$ is a soft int-field of A. ■

Proposition 3.4. *Let $\alpha : A \to A$ be an isomorphism and F be a soft int-field of A. Then $\alpha(F_K) = [\alpha(F)]_K$ for all $K \subseteq U$, where $[\alpha(F)]_K$ denotes the K-inclusion subset of $\alpha(F)$.*

Proof. Suppose that $K \subseteq F(0)$. Then $a \in \alpha(F_K) \Leftrightarrow \exists b \in F_K$ such that $\alpha(b) = a \Leftrightarrow F(b) \supseteq K$ such that $\alpha(b) = a \Leftrightarrow \alpha(F)(a) \supseteq K \Leftrightarrow a \in [\alpha(F)]_K$. Hence $\alpha(F_K) = [\alpha(F)]_K$ for all $K \subseteq F(0)$. Now, let $F(0) \subset K \subseteq U$. Since $F(0) \supseteq F(a)$ for all $a \in A$, so $F_K = \{a \in A : F(a) \supseteq K\} = \emptyset$ and hence $\alpha(F_K) = \emptyset$. Similarly, $[\alpha(F)]_K = \emptyset$. Therefore $\alpha(F_K) = [\alpha(F)]_K$ for all $K \subseteq U$. ■

We consider the definition of soft normal int-field over a field as follows:

Definition 3.5. *Let A/B be a normal field extension and $F \in (A/B)^{\clubsuit}$. Then F is said to be soft normal over B if $\alpha(F) \tilde{\subseteq} F$, for all $\alpha \in G(A/B)$.*

Theorem 3.6. *Let A/B be a normal field extension and $F \in (A/B)^{\clubsuit}$. Then F is soft normal over B if and only if F_K is normal field extension of B for all $K \in Im(F)$.*

Proof. Since $F \in (A/B)^{\clubsuit}$, F is soft int-field of A such that $F_K \supseteq B$, for all $K \in Im(F)$. Hence by Theorem 2.13, F_K is field extension of B for all $K \in Im(F)$. Then

$$\begin{aligned} & F \text{ is soft normal over } B \\ \Leftrightarrow\ & \alpha(F) \tilde{\subseteq} F, \quad \text{for all } \alpha \in G(A/B), \quad \text{(by Definition 3.5)} \\ \Leftrightarrow\ & [\alpha(F)]_K \subseteq F_K \quad \text{for all } K \in Im(F), \quad \text{for all } \alpha \in G(A/B) \\ \Leftrightarrow\ & \alpha(F_K) \subseteq F_K \quad \text{for all } K \in Im(F), \alpha \in G(A/B), \quad \text{(by Proposition 3.4)} \\ \Leftrightarrow\ & F_K \text{ is normal field extension of } B, \forall K \in Im(F), \quad \text{(by Theorem 2.15).} \end{aligned}$$

■

Example 3.7. Let $U = S_3$, the set of all permutations on the set $\{1, 2, 3\}$, be the universal set. Let $B = \mathbb{Q}$, the field of rational numbers and A be the splitting field of the irreducible polynomial $x^3 - 2$ over B. Hence $A = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)$ and A/B is a finite normal extension (see [20]). Intermediate fields of A/B are $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt{3}i), \mathbb{Q}(\frac{\sqrt[3]{2}}{2}(-1 + \sqrt{3}i)), \mathbb{Q}(\frac{\sqrt[3]{2}}{2}(-1 - \sqrt{3}i))$. We consider the chain of subfields of A as $B = \mathbb{Q} \subset \mathbb{Q}(\sqrt{3}i) \subset \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i) = A$.

Define a soft set F of A over S_3 as

$$F(a) = \begin{cases} S_3 & \text{if } \ a \in \mathbb{Q}, \\ A_3 & \text{if } \ a \in \mathbb{Q}(\sqrt{3}i) \setminus \mathbb{Q}, \\ I & \text{if } \ a \in \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i) \setminus \mathbb{Q}(\sqrt{3}i), \end{cases}$$

where A_3 is the set of all even permutations on $\{1, 2, 3\}$ and I is set containing only identity permutation on $\{1, 2, 3\}$. Then $I \subset A_3 \subset S_3$. Here $F_{S_3} = \mathbb{Q}, F_{A_3} = \mathbb{Q}(\sqrt{3}i), F_I = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)$. Then F_{S_3}, F_{A_3}, F_I are subfields of A and $F_K \supseteq B$, for all $K \in Im(F)$. Hence $F \in (A/B)^{\clubsuit}$.

Again, F_{S_3}, F_{A_3}, F_I are normal field extensions of B. Therefore by Theorem 3.6, F is soft normal over B. We consider the chain of subfields of A as $B = \mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i) = A$.

Define a soft set H of A over S_3 as

$$H(a) = \begin{cases} A_3 & \text{if } \ a \in \mathbb{Q}, \\ I & \text{if } \ a \in \mathbb{Q}(\sqrt[3]{2}) \setminus \mathbb{Q}, \\ \emptyset & \text{if } \ a \in \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i) \setminus \mathbb{Q}(\sqrt[3]{2}). \end{cases}$$

Hence $H_{A_3} = \mathbb{Q}, H_I = \mathbb{Q}(\sqrt[3]{2}), H_\emptyset = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)$. Therefore H is a soft int-field of A. Here $H_I = \mathbb{Q}(\sqrt[3]{2})$ is not normal extension of $\mathbb{Q}$. Hence by Theorem 3.6, H is not soft normal over B.

4. Soft Galois Correspondence

Definition 4.1. *Let A/B be a field extension. Let $a \in A$ and $\alpha \in G(A/B)$. Then the sets $G(A/B)^a$ and A^α are defined by*

$$G(A/B)^a = \{\alpha \in G(A/B) : \alpha(a) \neq a\},$$
$$A^\alpha = \{a \in A : \alpha(a) \neq a\}.$$

Proposition 4.2. *Let A/B be a field extension. Let $a, b \in A$ and $\alpha, \beta \in G(A/B)$. Then the following properties hold:*

(i) $G(A/B)^{a-b} \subseteq G(A/B)^a \cup G(A/B)^b$.

(ii) $G(A/B)^{ab^{-1}} \subseteq G(A/B)^a \cup G(A/B)^b$, *if $b \neq 0$, where 0 is the zero element of A.*

(iii) $A^{\alpha\beta^{-1}} \subseteq A^\alpha \cup A^\beta$.

(iv) $A^\alpha = \emptyset$ *if $\alpha : A \to A$ is an identity mapping.*

(v) $G(A/B)^b = \emptyset$, *for all $b \in B$.*

Proof. (i) Let $\alpha \notin G(A/B)^a \cup G(A/B)^b$. This implies $\alpha \notin G(A/B)^a$ and $\alpha \notin G(A/B)^b$. Then by Definition 4.1, $\alpha(a) = a$ and $\alpha(b) = b$. Since α is homomorphism, $\alpha(a - b) = \alpha(a) - \alpha(b) = a - b$. This shows that $\alpha \notin G(A/B)^{a-b}$. Therefore $G(A/B)^{a-b} \subseteq G(A/B)^a \cup G(A/B)^b$.

We omit the proofs of (ii)–(v), as proofs are similar. ∎

Definition 4.3. *Let A/B be a field extension and $F \in (A/B)^{\clubsuit}$. Define a soft subset $F^{'}$ of $G(A/B)$ induced by F as follows:*

$$F^{'}(\alpha) = U \setminus \bigcup_{a \in A^{\alpha}} F(a), \quad \textit{for all } \alpha \in G(A/B).$$

Theorem 4.4. *Let A/B be a field extension. If $F \in (A/B)^{\clubsuit}$ then $F^{'} \in G(A/B)^{\bigstar}$, where $F^{'}$ is defined in Definition 4.3.*

Proof. Let $\alpha, \beta \in G(A/B)$ and β^{-1} be the inverse of β in $G(A/B)$. Using part (iii) of Proposition 4.2, we have

$$\begin{aligned} F^{'}(\alpha\beta^{-1}) &= U \setminus \bigcup_{a \in A^{\alpha\beta^{-1}}} F(a) \supseteq U \setminus \bigcup_{a \in A^{\alpha} \cup A^{\beta}} F(a) \\ &= U \setminus \left[\left[\bigcup_{a \in A^{\alpha}} F(a) \right] \bigcup \left[\bigcup_{a \in A^{\beta}} F(a) \right] \right] \\ &= \left[U \setminus \bigcup_{a \in A^{\alpha}} F(a) \right] \bigcap \left[U \setminus \bigcup_{a \in A^{\beta}} F(a) \right] = F^{'}(\alpha) \cap F^{'}(\beta). \end{aligned}$$

Hence by Theorem 2.6, $F^{'}$ is a soft int-group of $G(A/B)$. Therefore $F^{'} \in G(A/B)^{\bigstar}$. ∎

Definition 4.5. *Let A/B be a field extension and $H \in G(A/B)^{\bigstar}$. Define a soft subset $H^{''}$ of A induced by H as follows:*

$$H^{''}(a) = U \setminus \bigcup_{\alpha \in G(A/B)^{a}} H(\alpha), \quad \textit{for all } a \in A.$$

Theorem 4.6. *Let A/B be a field extension. If $H \in G(A/B)^{\bigstar}$ then $H^{''} \in (A/B)^{\clubsuit}$, where $H^{''}$ is defined in Definition 4.5.*

Proof. Let $a, b \in A$. Now, we have

$$\begin{aligned} H^{''}(a-b) &= U \setminus \bigcup_{\alpha \in G(A/B)^{a-b}} H(\alpha) \\ &\supseteq U \setminus \bigcup_{\alpha \in G(A/B)^{a} \cup G(A/B)^{b}} H(\alpha) \quad \text{(by Proposition 4.2))} \\ &= U \setminus \left[\left[\bigcup_{\alpha \in G(A/B)^{a}} H(\alpha) \right] \bigcup \left[\bigcup_{\alpha \in G(A/B)^{b}} H(\alpha) \right] \right] \end{aligned}$$

$$= \left[U \setminus \bigcup_{\alpha \in G(A/B)^a} H(\alpha) \right] \bigcap \left[U \setminus \bigcup_{\alpha \in G(A/B)^b} H(\alpha) \right]$$
$$= H^{''}(a) \cap H^{''}(b).$$

Similarly, we shall get $H^{''}(ab^{-1}) \supseteq H^{''}(a) \cap H^{''}(b)$ for $a, b (\neq 0) \in A$, where 0 is the zero element of A. Hence by Definition 2.11, $H^{''}$ is a soft int-field of A.

Now, we shall prove that $H^{''}_K \supseteq B$ for all $K \in Im(H^{''})$.

Let $b \in B$. Then $\alpha(b) = b$, for all $\alpha \in G(A/B)$. So, $G(A/B)^b = \emptyset$. Then by Definition 4.5, we have $H^{''}(b) = U \setminus \bigcup_{\alpha \in G(A/B)^b} H(\alpha) = U \setminus \emptyset = U$. Then

$$\begin{aligned} b \in B &\Rightarrow H^{''}(b) = U \supseteq K, \quad \text{for all } K \in Im(H^{''}) \\ &\Rightarrow b \in H^{''}_K, \quad \text{for all } K \in Im(H^{''}). \end{aligned}$$

Hence $H^{''}_K \supseteq B$ for all $K \in Im(H^{''})$. Therefore $H^{''} \in (A/B)^{\clubsuit}$. ■

Proposition 4.7. *Let A/B be a field extension. Then for all $F_1, F_2 \in (A/B)^{\clubsuit}$ and $H_1, H_2 \in G(A/B)^{\bigstar}$, the following properties hold:*

(i) $F_1 \widetilde{\subseteq} F_2 \Rightarrow F_1^{'} \widetilde{\supseteq} F_2^{'}$.
(ii) $H_1 \widetilde{\subseteq} H_2 \Rightarrow H_1^{''} \widetilde{\supseteq} H_2^{''}$.
(iii) $(F_1^{'})^{''} \widetilde{\supseteq} F_1$.
(iv) $(H_1^{''})^{'} \widetilde{\supseteq} H_1$.
(v) $[(F_1^{'})^{''}]^{'} = F_1^{'}$.
(vi) $[(H_1^{''})^{'}]^{''} = H_1^{''}$.

Proof. (i) From Theorem 4.4, we have $F_1^{'}, F_2^{'} \in G(A/B)^{\bigstar}$. Let $\alpha \in G(A/B)$. Then

$$\begin{aligned} F_1 \widetilde{\subseteq} F_2 &\Rightarrow \bigcup_{a \in A^\alpha} F_1(a) \subseteq \bigcup_{a \in A^\alpha} F_2(a) \\ &\Rightarrow U \setminus \bigcup_{a \in A^\alpha} F_1(a) \supseteq U \setminus \bigcup_{a \in A^\alpha} F_2(a) \\ &\Rightarrow F_1^{'}(\alpha) \supseteq F_2^{'}(\alpha) \\ &\Rightarrow F_1^{'} \widetilde{\supseteq} F_2^{'} [\text{since} \alpha \in G(A/B) \text{is arbitrary}]. \end{aligned}$$

(ii) Proof is similar to that of (i)

(iii) By Theorems 4.4 and 4.6, we have $F_1 \in (A/B)^{\clubsuit} \Rightarrow F_1^{'} \in G(A/B)^{\bigstar} \Rightarrow (F_1^{'})^{''} \in (A/B)^{\clubsuit}$.

Now, for $a \in A$, we have

$$\begin{aligned}(F_1^{'})^{''}(a) &= U \setminus \bigcup_{\alpha \in G(A/B)^a} F_1^{'}(\alpha) \\ &= U \setminus \bigcup_{\alpha \in G(A/B)^a} [U \setminus \bigcup_{b \in A^\alpha} F_1(b)] \\ &= \bigcap_{\alpha \in G(A/B)^a} [\bigcup_{b \in A^\alpha} F_1(b)].\end{aligned}$$

If $G(A/B)^a = \emptyset$ then $(F_1^{'})^{''}(a) = U \supseteq F_1(a)$. If $G(A/B)^a \neq \emptyset$ then $a \in A^\alpha$ for $\alpha \in G(A/B)^a$. Hence, in this case, we have

$$\begin{aligned}(F_1^{'})^{''}(a) &= \bigcap_{\alpha \in G(A/B)^a} [\bigcup_{b \in A^\alpha} F_1(b)] \\ &\supseteq \bigcap_{\alpha \in G(A/B)^a} F_1(a) = F_1(a).\end{aligned}$$

Since $a \in A$ is arbitrary, $(F_1^{'})^{''} \widetilde{\supseteq} F_1$.

(iv) Proof is similar to that of (iii).

(v) Since $F_1 \in (A/B)^\clubsuit$, by Theorem 4.4, $F_1^{'} \in G(A/B)^\bigstar$. Hence by part (iv), we have $[(F_1^{'})^{''}]^{'} \widetilde{\supseteq} F_1^{'}$. Again by part (iii), $F_1 \widetilde{\subseteq} (F_1^{'})^{''}$. Then by using part (i), we have $[(F_1^{'})^{''}]^{'} \widetilde{\subseteq} F_1^{'}$. Therefore $[(F_1^{'})^{''}]^{'} = F_1^{'}$.

(vi) Proof is similar to that of (v). ■

Theorem 4.8. *Let A/B be a finite field extension and $F \in (A/B)^\clubsuit$. Let $Im(F) = \{K_1, K_2, \cdots, K_n\}$ such that $\emptyset \subseteq K_1 \subset K_2 \subset \cdots \subset K_n \subseteq U$, where $n \in \mathbb{N}$. Then the soft subset $F^{'}$ of $G(A/B)$, as defined in Definition 4.3, is given by*

$$F^{'}(\alpha) = \begin{cases} U & \text{if } \alpha \in G(A/F_{K_1}), \\ U \setminus K_i & \text{if } \alpha \in G(A/F_{K_{i+1}}) \setminus G(A/F_{K_i}), i = 1, 2, \cdots, n-1, \\ U \setminus K_n & \text{if } \alpha \in G(A/B) \setminus G(A/F_{K_n}). \end{cases}$$

Proof. Since $F \in (A/B)^\clubsuit$, F is a soft int-field of A such that $F_K \supseteq B$, for all $K \in Im(F)$. Since $K_1 \subset K_2 \subset \cdots \subset K_n$, $A = F_{K_1} \supset F_{K_2} \supset \cdots \supset F_{K_n} \supseteq B$, where $F_{K_i} (1 \leq i \leq n, n \in \mathbb{N})$ are inclusion subsets of F. Hence by crisp concept, $G(A/F_{K_1}) \subset G(A/F_{K_2}) \subset \cdots \subset G(A/F_{K_n}) \subseteq G(A/B)$.

Now $G(A/F_{K_1}) = G(A/A) = \{\sigma \in G(A/B) : \sigma(a) = a, \text{ for all } a \in A\} = \{id : A \to A \text{an identity map}\}$. If $\alpha \in G(A/F_{K_1})$ then $\alpha = id$ (an identity map from A to A). Then by Definition 4.3, $F^{'}(\alpha) = F^{'}(id) = U \setminus \bigcup_{a \in A^{id}} F(a)$, where $A^{id} = \{a \in A : id(a) \neq a\} = \emptyset$. Then by Definition 4.3, $\bigcup_{a \in A^{id}} F(a) = \emptyset$. Hence for $\alpha \in G(A/F_{K_1}), F^{'}(\alpha) = U$.

If $\alpha \in G(A/F_{K_2}) \setminus G(A/F_{K_1})$ then $\alpha \in G(A/F_{K_2})$ but $\alpha \notin G(A/F_{K_1})$. Now, $\alpha \in G(A/F_{K_2}) \Rightarrow \alpha(a) = a$, for all $a \in F_{K_2} \Rightarrow a \notin A^\alpha$, for all $a \in F_{K_2}$.

Again, we have

$$\begin{aligned}\alpha \notin G(A/F_{K_1}) &\Rightarrow \exists b \in F_{K_1} \quad \text{such that} \quad \alpha(b) \neq b\\ &\Rightarrow \exists b \in F_{K_1} \quad \text{such that} \quad b \in A^{\alpha}.\end{aligned}$$

Then $\exists b \in F_{K_1} \setminus F_{K_2}$ such that $b \in A^{\alpha}$. Hence for $b \in A^{\alpha}$, we have $K_2 \supset F(b) \supseteq K_1$. Since $Im(F) = \{K_1, K_2, \cdots K_n\}$, $F(b) = K_1$ for $b \in A^{\alpha}$. Hence for $\alpha \in G(A/F_{K_2}) \setminus G(A/F_{K_1})$,

$$F^{'}(\alpha) = U \setminus \bigcup_{a \in A^{\alpha}} F(a) = U \setminus K_1.$$

Proceeding in this way, we have $F^{'}(\alpha) = U \setminus K_i$ for all $\alpha \in G(A/F_{K_{i+1}}) \setminus G(A/F_{K_i})$, where $1 \leq i \leq n-1, n \in \mathbb{N}$. If $\alpha \in G(A/B) \setminus G(A/F_{K_n})$ then $\alpha \in G(A/B)$ but $\alpha \notin G(A/F_{K_n})$. Now, we have

$$\alpha \in G(A/B) \Rightarrow \alpha(a) = a \Rightarrow a \notin A^{\alpha},$$

for all $a \in B$.

Again, we have

$$\begin{aligned}\alpha \notin G(A/F_{K_n}) &\Rightarrow \exists c \in F_{K_n} \quad \text{such that} \quad \alpha(c) \neq c\\ &\Rightarrow \exists c \in F_{K_n} \quad \text{such that} \quad c \in A^{\alpha}.\end{aligned}$$

Then there exists $c \in F_{K_n} \setminus B$ such that $c \in A^{\alpha}$. Hence for $c \in A^{\alpha}$, we have $F(c) \supseteq K_n$. This implies $F(c) = K_n$. Therefore

$$F^{'}(\alpha) = U \setminus \bigcup_{a \in A^{\alpha}} F(a) = U \setminus K_n.$$

■

Theorem 4.9. *Let A/B be a finite field extension and $H \in G(A/B)^{\star}$. Let $Im(H) = \{L_1, L_2, \cdots, L_m\}$ such that $\emptyset \subseteq L_1 \subset L_2 \subset \cdots \subset L_m \subseteq U$, where $m \in \mathbb{N}$. Then the soft subset $H^{''}$ of A, as defined in Definition 4.5, is given by*

$$H^{''}(a) = \begin{cases} U & \text{if } \ a \in A^{H_{L_1}}, \\ U \setminus L_j & \text{if } \ a \in A^{H_{L_{j+1}}} \setminus A^{H_{L_j}}, j = 1, 2, \cdots, m-1, \\ U \setminus L_m & \text{if } \ a \in A \setminus A^{H_{L_m}}. \end{cases}$$

Proof. Since $H \in G(A/B)^{\star}$, H is a soft int-group of $G(A/B)$. Since $L_1 \subset L_2 \subset \cdots \subset L_m$, $G(A/B) = H_{L_1} \supset H_{L_2} \supset \cdots \supset H_{L_m}$, where $H_{L_j} (1 \leq j \leq m, m \in \mathbb{N})$ are inclusion subsets of H. Also, each H_{L_j} are subgroups of $G(A/B)$. Hence by Proposition 2.16, $A^{H_{L_1}} \subset A^{H_{L_2}} \subset \cdots \subset A^{H_{L_m}} \subseteq A$. If $a \in A^{H_{L_1}}$ then $\alpha(a) = a$ for all $\alpha \in H_{L_1} = G(A/B)$. Hence by Definition 4.5, $H^{''}(a) = U \setminus \bigcup_{\alpha \in G(A/B)^a} H(\alpha)$, where $G(A/B)^a = \{\alpha \in G(A/B) : \alpha(a) \neq a\} = \emptyset$. Hence $\bigcup_{\alpha \in G(A/B)^a} H(\alpha) = \emptyset$. Therefore $H^{''}(a) = U \setminus \emptyset = U$.

If $a \in A^{H_{L_2}} \setminus A^{H_{L_1}}$ then $a \in A^{H_{L_2}}$ but $a \notin A^{H_{L_1}}$. Now, we have

$$\begin{aligned} a \in A^{H_{L_2}} &\Rightarrow \alpha(a) = a, \quad \text{for all } \alpha \in H_{L_2} \\ &\Rightarrow \alpha \notin G(A/B)^a, \quad \text{for all } \alpha \in H_{L_2}. \end{aligned}$$

Also, we have

$$\begin{aligned} a \notin A^{H_{L_1}} &\Rightarrow \exists \delta \in H_{L_1} \quad \text{such that} \quad \delta(a) \neq a \\ &\Rightarrow \exists \delta \in H_{L_1} \quad \text{such that} \quad \delta \in G(A/B)^a. \end{aligned}$$

Then there exists $\delta \in H_{L_1} \setminus H_{L_2}$ such that $\delta \in G(A/B)^a$. Hence for $\delta \in G(A/B)^a$, we have $L_2 \supset H(\delta) \supseteq L_1$. Since $Im(H) = \{L_1, L_2, \cdots, L_m\}$, $H(\delta) = L_1$ for $\delta \in G(A/B)^a$. Therefore $H^{''}(a) = U \setminus \bigcup_{\alpha \in G(A/B)^a} H(\alpha) = U \setminus L_1$.

Proceeding in this way, we have $H^{''}(a) = U \setminus L_j$ for all $a \in A^{H_{L_{j+1}}} \setminus A^{H_{L_j}}$, where $1 \leq j \leq m-1, m \in \mathbb{N}$.

If $a \in A \setminus A^{H_{L_m}}$ then $a \in A$ but $a \notin A^{H_{L_m}}$. Now, we have

$$\begin{aligned} a \notin A^{H_{L_m}} &\Rightarrow \exists \gamma \in H_{L_m} \quad \text{such that} \quad \gamma(a) \neq a \\ &\Rightarrow \exists \gamma \in H_{L_m} \quad \text{such that} \quad \gamma \in G(A/B)^a \\ &\Rightarrow \exists \gamma \in G(A/B) \text{with} H(\gamma) = L_m \quad \text{such that} \quad \gamma \in G(A/B)^a. \end{aligned}$$

Therefore $H^{''}(a) = U \setminus \bigcup_{\alpha \in G(A/B)^a} H(\alpha) = U \setminus L_m$. ∎

Example 4.10. Let $B = \mathbb{Q}$, the field of rational numbers and $A = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Then A/B is a Galois extension and corresponding Galois group $G(A/B)$ is the Klein's $4-$group of elements $\{id, \alpha, \beta, \gamma\}$, where $id, \alpha, \beta, \gamma$ are $B-$automorphisms of A such that

$$\begin{aligned} &id : A \to A \quad \text{is the identity automorphism;} \\ &\alpha(\sqrt{3}) = \sqrt{3}, \alpha(\sqrt{5}) = -\sqrt{5}; \\ &\beta(\sqrt{3}) = -\sqrt{3}, \beta(\sqrt{5}) = -\sqrt{5}; \\ &\gamma(\sqrt{3}) = -\sqrt{3}, \gamma(\sqrt{5}) = \sqrt{5}. \end{aligned}$$

Subgroups of $G(A/B)$ are $\{id\}, \{id, \alpha\}, \{id, \beta\}, \{id, \gamma\}, G(A/B)$. We consider the chain of subgroups $\{id\} \subset \{id, \alpha\} \subset G(A/B)$.

Define a soft int-group H of $G(A/B)$ over U as

$$H(\sigma) = \begin{cases} U & \text{if } \sigma = id, \\ L_1 & \text{if } \sigma \in \{id, \alpha\} \setminus \{id\}, \\ L_2 & \text{if } \sigma \in G(A/B) \setminus \{id, \alpha\}, \end{cases}$$

where $L_2 \subset L_1 \subset U$.. Therefore $H_U \subset H_{L_1} \subset H_{L_2}$ is the chain of inclusion subgroups of H, where $H_U = \{id\}, H_{L_1} = \{id, \alpha\}, H_{L_2} = G(A/B)$. Hence corresponding chain of fixed fields is $A^{H_U} \supset A^{H_{L_1}} \supset A^{H_{L_2}}$, where $A^{H_U} = \{a \in$

$A : \alpha(a) = a$, for all $\alpha \in H_U\} = A = \mathbb{Q}(\sqrt{3}, \sqrt{5})$, $A^{H_{L_1}} = \{a \in A : \alpha(a) = a$, for all $\alpha \in H_{L_1}\} = \mathbb{Q}(\sqrt{3})$ (see [16]), $A^{H_{L_2}} = \{a \in A : \alpha(a) = a$, for all $\alpha \in H_{L_2}\} = B = \mathbb{Q}$.

Hence from Theorem 4.9, the soft int-field $H^{''} \in (A/B)^{\clubsuit}$ corresponding to H is as follows:

$$H^{''}(a) = \begin{cases} U & \text{if } a \in A^{H_{L_2}} = \mathbb{Q}, \\ U \setminus L_2 & \text{if } a \in A^{H_{L_1}} \setminus A^{H_{L_2}} = \mathbb{Q}(\sqrt{3}) \setminus \mathbb{Q}, \\ U \setminus L_1 & \text{if } a \in A^{H_U} \setminus A^{H_{L_1}} = \mathbb{Q}(\sqrt{3}, \sqrt{5}) \setminus \mathbb{Q}(\sqrt{3}). \end{cases}$$

Therefore the inclusion subfields of $H^{''}$ are $H^{''}_U = \mathbb{Q}, H^{''}_{U\setminus L_2} = \mathbb{Q}(\sqrt{3}), H^{''}_{U\setminus L_1} = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ and the chain of inclusion subfields of $H^{''}$ is $H^{''}_U \subset H^{''}_{U\setminus L_2} \subset H^{''}_{U\setminus L_1}$.

Then by crisp concept, we get $G(A/H^{''}_U) = \{\sigma \in G(A/B) : \sigma(a) = a$, for all $a \in H^{''}_U = \mathbb{Q}\} = \{id, \alpha, \beta, \gamma\} = G(A/B)$.

Similarly, $G(A/H^{''}_{U\setminus L_2}) = G(A/\mathbb{Q}(\sqrt{3})) = \{id, \alpha\}$ and $G(A/H^{''}_{U\setminus L_1}) = G(A/\mathbb{Q}(\sqrt{3}, \sqrt{5})) = \{id\}$. Therefore $G(A/H^{''}_{U\setminus L_1}) \subset G(A/H^{''}_{U\setminus L_2}) \subset G(A/H^{''}_U)$. Hence from Theorem 4.8, the soft int-group $(H^{''})^{'}$ of $G(A/B)$ is as follows:

$$(H^{''})^{'}(\sigma) = \begin{cases} U & \text{if } \sigma \in G(A/H^{''}_{U\setminus L_1}), \\ U \setminus (U \setminus L_1) = L_1 & \text{if } \sigma \in G(A/H^{''}_{U\setminus L_2}) \setminus G(A/H^{''}_{U\setminus L_1}), \\ U \setminus (U \setminus L_2) = L_2 & \text{if } \sigma \in G(A/H^{''}_U) \setminus G(A/H^{''}_{U\setminus L_2}). \end{cases}$$

Therefore $(H^{''})^{'} = H$.

Proposition 4.11. *Let A/B be a field extension, $F \in (A/B)^{\clubsuit}$ and $H \in G(A/B)^{\bigstar}$. Then*

(i) $F^{'}_{U\setminus K} = G(A/F_{K^\star})$,
(ii) $H^{''}_{U\setminus L} = A^{H_{L^\star}}$,
(iii) $F^{'}_{(U\setminus K)^\star} = G(A/F_K)$,
(iv) $H^{''}_{(U\setminus L)^\star} = A^{H_L}$,

where $K, L \subseteq U, F_{K^\star} = \{a \in A : F(a) \supset K\}, H_{L^\star} = \{\alpha \in G(A/B) : H(\alpha) \supset L\}$, $F^{'}_{(U\setminus K)^\star} = \{\alpha \in G(A/B) : F^{'}(\alpha) \supset U \setminus K\}$ and $H^{''}_{(U\setminus L)^\star} = \{a \in A : H^{''}(a) \supset U \setminus L\}$.

Proof. (i) By Theorem 4.4, we have $F^{'} \in G(A/B)^{\bigstar}$, i.e. $F^{'}$ is a soft int-group of $G(A/B)$. Now,

$$\begin{aligned}\alpha \in F'_{U\setminus K} &\Leftrightarrow F'(\alpha) \supseteq U \setminus K\\ &\Leftrightarrow U \setminus \bigcup_{a\in A^{\alpha}} F(a) \supseteq U \setminus K, \quad \text{(by Definition 4.3)}\\ &\Leftrightarrow \bigcup_{a\in A^{\alpha}} F(a) \subseteq K\\ &\Leftrightarrow F(a) \subseteq K, \quad \text{for all } a \in A^{\alpha}\\ &\Leftrightarrow F(a) \subseteq K \quad \text{for} \quad a \in A \quad \text{such that} \quad \alpha(a) \neq a.\end{aligned}$$

Therefore $a \in A, \alpha(a) \neq a \Rightarrow F(a) \subseteq K$.

Since contrapositive of a true statement is also true, we have

$$a \in A, F(a) \supset K \Rightarrow \alpha(a) = a.$$

Hence $a \in F_{K^\star} \Rightarrow \alpha(a) = a$. This implies $\alpha \in G(A/F_{K^\star})$. Therefore $F'_{U\setminus K} \subseteq G(A/F_{K^\star})$.

Similarly, it can be shown that $G(A/F_{K^\star}) \subseteq F'_{U\setminus K}$. Therefore $F'_{U\setminus K} = G(A/F_{K^\star})$. Proofs of (ii)–(iv) are similar to that of (i). ∎

Theorem 4.12. *Let A/B be a finite Galois extension. Then the following statements hold:*

- (i) *Define a pair of maps (f, h), where $f : (A/B)^{\clubsuit} \to G(A/B)^{\bigstar}$ is given by $f(F) = F'$, for all $F \in (A/B)^{\clubsuit}$ and $h : G(A/B)^{\bigstar} \to (A/B)^{\clubsuit}$ is given by $h(H) = H''$, for all $H \in G(A/B)^{\bigstar}$. Then f, h are both bijective inclusion reversing correspondence between $(A/B)^{\clubsuit}$ and $G(A/B)^{\bigstar}$ such that f, h are inverses of one another.*
- (ii) *If H is a soft normal int-group of $G(A/B)$ then $[G(A/B) : H] = [A^{H_L} : B]$, where $L = H(id)$ and $id \in G(A/B)$ is the identity map.*
- (iii) *Let $F \in (A/B)^{\clubsuit}$. Then F is soft normal over B if and only if F' is soft normal int-group of $G(A/B)$.*
- (iv) *Let $F \in (A/B)^{\clubsuit}$ such that F is soft normal over B. If $Im(F) = \{K_1, K_2, \cdots, K_n\}$ such that $K_1 \subset K_2 \subset \cdots \subset K_n$, where $n \in \mathbb{N}$, then $[G(A/B) : F'] = [F_{K_1} : B]$.*
  Moreover, $G(A/B)/F'$ is isomorphic to $G(F_{K_1}/B)$.

Proof. (i) It is clear from Theorems 4.4 and 4.6 that the maps f, h are well-defined. Let $F_1, F_2 \in (A/B)^{\clubsuit}$. Then by Proposition 4.7,

$$F_1 \widetilde{\subseteq} F_2 \Rightarrow F'_1 \widetilde{\supseteq} F'_2, \quad \text{i.e.} \quad f(F_1) \widetilde{\supseteq} f(F_2).$$

Now, let $H_1, H_2 \in G(A/B)^{\bigstar}$. Again by Proposition 4.7,

$$H_1 \widetilde{\subseteq} H_2 \Rightarrow H''_1 \widetilde{\supseteq} H''_2, \quad \text{i.e.} \quad h(H_1) \widetilde{\supseteq} h(H_2).$$

Therefore the maps f, h are inclusion reversing.

To prove f, h are inverses of one another, it is only to prove that $hf(F) = F$, for all $F \in (A/B)^{\clubsuit}$ and $fh(H) = H$, for all $H \in G(A/B)^{\bigstar}$. Let $F \in (A/B)^{\clubsuit}$. Then $f(F) = F^{'} \in G(A/B)^{\bigstar}$. Hence $hf(F) = h(f(F)) = h(F^{'}) = (F^{'})^{''} \in (A/B)^{\clubsuit}$. By part (iii) of Proposition 4.7, we have

$$hf(F)\widetilde{\supseteq}F. \tag{2}$$

Now, we shall prove that

$$[hf(F)]_K = F_K \quad \text{for any} \quad K \subseteq U. \tag{3}$$

Using Fundamental Theorem of Galois Theory and Proposition 4.11, we have $[hf(F)]_K = [(F^{'})^{''}]_K = A^{F^{'}_{(U\setminus K)^{\star}}} = A^{G(A/F_K)} = F_K$.

Suppose there exists $a \in A$ such that $hf(F)(a) \supset F(a)$. Let $hf(F)(a) = L$ and $F(a) = M$. Hence $L \supset M$. Now, we have

$$\begin{aligned} hf(F)(a) = L &\Rightarrow a \in hf(F)_L = F_L \quad \text{(by Eq. (3))} \\ &\Rightarrow F(a) \supseteq L \Rightarrow M \supseteq L. \end{aligned}$$

This is a contradiction. Hence $hf(F)(a) \not\supset F(a)$, for all $a \in A$. Therefore from Eq. (2), we get $hf(F) = F$.

Similarly, we can prove that $fh(H) = H$, for all $H \in G(A/B)^{\bigstar}$. From this, it also follows that f, h are both bijective maps.

(ii) Let H be a soft normal int-group of the Galois group $G(A/B)$. Then by Theorem 2.10, $G(A/B)/H$ forms a group and $G(A/B)/H$ is isomorphic to the group $G(A/B)/H_L$, where $L = H(id)$ and $id \in G(A/B)$ is the identity mapping. Therefore $[G(A/B) : H] = [G(A/B) : H_L]$. Since H_L is a subgroup of $G(A/B)$, by Proposition 2.16, the fixed field of H_L is A^{H_L}. Also, since A/B is a Galois extension, the fixed field of $G(A/B)$ is B. Then by Fundamental Theorem of Galois Theory, we have $[G(A/B) : H_L] = [A^{H_L} : B]$. Therefore $[G(A/B) : H] = [A^{H_L} : B]$, where $L = H(id)$.

(iii) Let $F \in (A/B)^{\clubsuit}$. Since A/B is a finite Galois extension, the Galois group $G(A/B)$ is finite. Hence by Fundamental Theorem of Galois Theory there exist finite number of intermediate fields of A/B. If $Im(F)$ is infinite, then there will be infinitely many intermediate fields $F_K, K \in Im(F)$. This is a contradiction. Hence $Im(F)$ is finite. Suppose $Im(F) = \{K_1, K_2, \cdots, K_n\}$ such that $K_1 \subset K_2 \subset \cdots \subset K_n$, where $n \in \mathbb{N}$. Then

$$\begin{aligned} & F \;\text{ is soft normal over } B \\ \Leftrightarrow\; & F_{K_i} \;\text{ is normal field extension of }\; B, \forall K_i \in Im(F), \;\; \text{(by Theorem 3.6)} \\ \Leftrightarrow\; & G(A/F_{K_i}) \;\text{ is normal subgroup of }\; G(A/B), \;\; \text{for all } K_i \in Im(F), \\ & \quad \text{(by Fundamental Theorem of Galois Theory)} \\ \Leftrightarrow\; & F^{'} \;\text{ is soft normal int-group of }\; G(A/B), \;\; \text{by Theorem 2.8} \\ & \quad \text{and the fact that the inclusion subsets of } F^{'} \text{ are } G(A/B) \\ & \quad \text{and } G(A/F_{K_i}), \;\; \text{for all } K_i \in Im(F), \;\; \text{(by Theorem 4.8).} \end{aligned}$$

(iv) Let $Im(F) = \{K_1, K_2, \cdots, K_n\}$ such that $K_1 \subset K_2 \subset \cdots \subset K_n$. Since F is soft normal over B, by Part (iii), we have $F^{'}$ is soft normal int-group of $G(A/B)$. Hence by Part (ii), we get $[G(A/B) : F^{'}] = [A^{F^{'}_L} : B]$, where $L = F^{'}(id)$ and $id \in G(A/B)$ is the identity map. Since $K_1 \subset K_2 \subset \cdots \subset K_n$, $A = F_{K_1} \supset F_{K_2} \supset \cdots \supset F_{K_n}$. Hence $G(A/F_{K_1}) = G(A/A) = \{id\}$. By Theorem 4.8, $L = F^{'}(id) = U$ and $F^{'}_L = G(A/F_{K_1})$. Therefore by Fundamental Theorem of Galois Theory, we get $[A^{F^{'}_L} : B] = [A^{G(A/F_{K_1})} : B] = [F_{K_1} : B]$.

Again by Theorem 2.10, $G(A/B)/F^{'}$ is isomorphic to $G(A/B)/F^{'}_L$. Also, we have proved that $F^{'}_L = G(A/F_{K_1})$. Hence by Fundamental Theorem of Galois Theory, $G(A/B)/F^{'}_L = G(A/B)/G(A/F_{K_1}) \cong G(F_{K_1}/B)$. Therefore $G(A/B)/F^{'}$ is isomorphic to $G(F_{K_1}/B)$. ∎

Note 4.13. The correspondence which has been studied in Theorem 4.12 will be called *Soft Galois Correspondence*.

Example 4.14. In Example 4.10, A/B is a finite Galois extension and $H \in G(A/B)^{\star}$. We have already checked that $(H^{''})^{'} = H$, i.e., $fh(H) = H$. Since each inclusion subgroups of H, viz., H_U, H_{L_1}, H_{L_2} are normal subgroups of $G(A/B)$. Then by Theorem 2.8, H is a soft normal int-group of $G(A/B)$. By Theorem 4.12, we have $[G(A/B) : H] = [A^{H_U} : B] = [\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 4$.

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