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Nil-Extensions of Simple and π-Inverse Ordered Semigroups

Kalyan Hansda
Department of Mathematics, Visva-Bharati University, Santiniketan, Bolpur-731235, West Bengal, India
Email: kalyanh4@gmail.com

Amlan Jamadar
Department of Mathematics, Rampurhat College, Rampurhat-731224, West Bengal, India
Email: amlanjamadar@gmail.com

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Abstract. This paper is an attempt to study ordered semigroups which are nil-extensions of simple and π-inverse ordered semigroups. Different characterizations of complete semilattice decomposition of nil-extensions of ordered semigroups have been given here.

Keywords: l-Archimedean; π-Regular; Nil-Extension; Ordered idempotent; Simple ordered semigroup; π-Inverse ordered semigroup.

1. Introduction and Preliminaries

A semigroup $(S, \cdot)$ with an order relation $\leq$ is called an ordered semigroup if for all $a,\ b,\ x \in S,\ a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. Let $(S, \cdot, \leq)$ be an ordered semigroup. For a subset A of S, the downward closure of A is given by $(A] = \{x \in S : x \leq a,$ for some $a \in A\}$. An element a of S

is said to be regular (resp. intra-regular) if $a \in (aSa]$ (resp. $a \in (Sa^2S]$). We denote set of regular and intra-regular elements by $Reg_{\leq}(S)$ and $Intra_{\leq}(S)$ respectively. An element $b \in S$ is called ordered inverse [3] of a if $a \leq aba$ and $b \leq bab$. The set of all ordered inverses of an element $a \in S$ is denoted by $V_{\leq}(a)$. Throughout this paper, a', a'' are the ordered inverses of a unless otherwise stated. An element $e \in S$ is said to be ordered idempotent if $e \leq e^2$. The set of all ordered idempotents of S is denoted by $E_{\leq}(S)$.

An ordered semigroup S is called Archimedean [2] if for every $a, b \in S$ there is $m \in \mathbb{N}$ such that $b^m \in (SaS]$. S is called r(l *or* t)-Archimedean [2] if for every $a, b \in S$, there exists $m \in \mathbb{N}$ such that $b^m \in (aS]$ ($b^m \in (Sa]$ or $b^m \in (aSa]$).

A nonempty subset A of S is called a left (right) ideal of S, if $SA \subseteq A$ ($AS \subseteq A$) and $(A] = A$ (see [5]). A nonempty subset A is called a (two-sided) ideal of S if it is both a left and a right ideal of S. An left (right) ideal I of S is proper if $I \neq S$. S is left (right) simple if it does not contain proper left (right) ideals. An ordered semigroup S is called simple if for every ideal I of S, we have $I = S$. S is called t-simple if it is both left and right simple.

The principal [5] left ideal, right ideal, ideal and bi-ideal generated by $a \in S$ are denoted by $L(a)$, $R(a)$, $I(a)$ and $B(a)$ respectively and defined by

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } B(a) = (a \cup aSa].$$

Kehayopulu [5] defined Greens relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows:

$$a\mathcal{L}b\ if\ L(a) = L(b),\ a\mathcal{R}b \text{ if } R(a) = R(b),\ a\mathcal{J}b \text{ if } I(a) = I(b) \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

An ordered semigroup S is called π-regular (resp. intra π-regular) [2] if for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \in (a^mSa^m]$ (resp. $a^m \in (Sa^{2m}S]$). We denote set of all π-regular and intra π-regular elements by $\pi Reg_{\leq}(S)$ and $\Pi Intra_{\leq}(S)$ respectively. A π-regular ordered semigroup S is called π-inverse [4] if for every $a \in S$, there is $m \in \mathbb{N}$ such that any two inverses of a^m are $\mathcal{H}$-related.

Nil-extensions of an ordered semigroup S with zero 0 are precisely the ideal extensions of an ideal I of S by the nilpotent ordered semigroup S/I [6]. The theory of nil-extensions in ordered semigroup have been studied by Cao and Xu [2], Kehayopulu and Tsingelis [7], Bhuniya and Hansda [1] and many others. Cao and Xu [2] studied ordered semigroups which are nil-extensions of t-simple ordered semigroups. These ordered semigroups are natural generalization of π-groups. Sadhya and Hansda [8] studied these ordered semigroups under the name of π-t-simple ordered semigroups.

The aim of this work is to describe nil-extensions of π-inverse, left π-inverse ordered semigroups. Our approach allows one to see the role of ordered inverses of an ordered semigroup in this characterization. Furthermore, complete semilattice decompositions of the nil-extensions of π-inverse, left π-inverse ordered semigroups have been given here.

A congruence ρ on S is called semilattice if for all $a, b \in S$ $a \ \rho \ a^2$ and $ab\rho ba$. A semilattice congruence ρ on S is called complete if $a \leq b$ implies $a\rho ab$. The ordered semigroup S is called complete semilattice of subsemigroup of type τ if there exists a complete semilattice congruence ρ such that $(x)_\rho$ is a type τ subsemigroup of S. Equivalently, there exists a semilattice Y and a family of subsemigroups $\{S\}_{\alpha \in Y}$ of type τ of S such that:

(i) $S_\alpha \cap S_\beta = \phi$ for any $\alpha, \ \beta \in Y$ with $\alpha \neq \beta$,
(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,
(iii) $S_\alpha S_\beta \subseteq S_{\alpha \beta}$ for any $\alpha, \ \beta \in Y$,
(iv) $S_\beta \cap (S_\alpha] \neq \phi$ implies $\beta \preceq \alpha$, where $\preceq$ is the order of the semilattice Y defined by $\preceq := \{(\alpha, \ \beta) \mid \alpha = \alpha \, \beta \, (\beta \, \alpha)\}$ (see [7]).

Let S be a π-regular ordered semigroup. Due to Sadhya and Hansda [9], the following equivalence relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{J}^*$ and $\mathcal{H}^*$ are given by:

$$\begin{aligned} a\mathcal{L}^*b &\Leftrightarrow a^m\mathcal{L}b^n, \\ a\mathcal{R}^*b &\Leftrightarrow a^m\mathcal{R}b^n, \\ a\mathcal{J}^*b &\Leftrightarrow a^m\mathcal{J}b^n, \\ \mathcal{H}^* &= \mathcal{L}^* \cap \mathcal{R}^*, \end{aligned}$$

where $a, b \in S$ and m, n are the smallest positive integers such that $a^m, b^n \in Reg_{\leq}(S)$.

For $a, b \in S$, $a|b$ if and only if there exist $x, y \in S^1$ such that $b \leq xay$.

For the sake of convenience of general reader we state following results.

Theorem 1.1. [4, Theorem 2.3] *The following conditions are equivalent on an ordered semigroup S:*

(i) *S is a π-inverse ordered semigroup;*
(ii) *S is π-regular and for every $e, f \in E_{\leq}(S)$, there is $m \in \mathbb{N}$ such that $(ef)^m \in (fSe]$;*
(iii) *S is π-regular and for every $e, f \in E_{\leq}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$.*

Theorem 1.2. [2, Theorem 3.5] *The following conditions are equivalent on a po-semigroup S:*

(i) *S is a nil-extension of a simple po-semigroup;*
(ii) *S is an Archimedean po-semigroup in which $Intra(S) \neq \phi$.*

Corollary 1.3. [2, Corollary 5.2] *The following conditions are equivalent on a po-semigroup S:*

(i) *S is a nil-extension of a t-simple po-semigroup;*
(ii) *S is a t-Archimedean po-semigroup in which $Intra(S) \neq \phi$.*

2. Nil-Extensions of Simple and π-Inverse Ordered Semigroups

This section is aiming to characterize all ordered semigroups which are nil-extensions of inverse, simple and π-inverse, left simple and π-inverse ordered semigroups. We define the sets $\mathbf{V}_{\leq}(S)$ and $\Pi\mathbf{V}_{\leq}(S)$ as follows:

$$\mathbf{V}_{\leq}(S) = \{a \in S \mid \text{for any } x,\ y \in V_{\leq}(a) \text{ implies } x\mathcal{H}y\},$$
$$\Pi\mathbf{V}_{\leq}(S) = \{a \in S\ (\exists m \in \mathbb{N}) \mid \text{for any } x,\ y \in V_{\leq}(a^m) \text{ implies } x\mathcal{H}y\}.$$

Lemma 2.1. *Let S be an ordered semigroup. Then the following conditions are equivalent on S:*
(i) *For every $a \in S$ and $c \in \boldsymbol{V}_{\leq}(S)$, $a \mid c$ implies $a^2 \mid c$;*
(ii) *for every $a, b \in S$ and $c \in \boldsymbol{V}_{\leq}(S)$, $a \mid c$ and $b \mid c$ implies $ab \mid c$.*

Proof. (i)$\Rightarrow$(ii): Let $a, b \in S$ and $c \in \mathbf{V}_{\leq}(S)$ be such that $a \mid c$ and $b \mid c$. Then there are $x, y, z, w \in S$ such that $c \leq xay$ and $c \leq zbw$. Now $c \in \mathbf{V}_{\leq}(S)$ implies that there exists $t \in S$ such that $c \leq ctc \leq zbwtxay$. Thus $bwtxa \mid c$, and so by the given condition $(bwtxa)^2 \mid c$. That is, $c \in (SbwtxabwtxaS] \subseteq (SabS]$. Hence $ab \mid c$.

(ii)$\Rightarrow$(i): This is obvious. ∎

Theorem 2.2. *Let an ordered semigroup S be a complete semilattice Y of subsemigroups $\{S_\alpha\}_{\alpha \in Y}$. Then the following statements hold:*
(i) $\boldsymbol{V}_{\leq}(S) = \cup_{\alpha \in Y} \boldsymbol{V}_{\leq}(S_\alpha)$.
(ii) *S is inverse if and only if S_α is inverse for all $\alpha \in Y$.*
(iii) *S is π-inverse if and only if S_α is π-inverse for all $\alpha \in Y$.*

Proof. (i): It is obvious that $\mathbf{V}_{\leq}(S) \supseteq \cup_{\alpha \in Y} \mathbf{V}_{\leq}(S_\alpha)$. Let $a \in S_\alpha \cap \mathbf{V}_{\leq}(S)$. Then exist $x \in S_\beta$, $y \in S_\gamma$ such that for any $x, y \in V_{\leq}(a)$ implies $x\mathcal{H}y$. Now $a \leq axa$ implies that $\alpha \leq \alpha\beta\alpha$. From $\alpha\beta\alpha \leq \alpha\beta \leq \alpha$, we obtained $\alpha = \alpha\beta$. Now $x \leq xax$ implies $\beta \leq \beta\alpha\beta$, which implies that $\alpha = \alpha\beta = \beta\alpha = \beta$. Hence $x \in S_\alpha$. Similarly $y \in S_\alpha$. Thus $a \in \mathbf{V}_{\leq}(S_\alpha)$. Therefore we have $\mathbf{V}_{\leq}(S) = (\cup_{\alpha \in Y} \mathbf{V}_{\leq}(S_\alpha)) \cap \mathbf{V}_{\leq}(S) \subseteq \cup_{\alpha \in Y} \mathbf{V}_{\leq}(S_\alpha)$. Therefore, the first equality holds.

Both (ii) and (iii) are immediate consequences of (i). ∎

Lemma 2.3. *Let an ordered semigroup S be a nil-extension of a semigroup K of type τ and $\mathbf{V}_{\leq}(S) \neq \phi$. Then the following statements hold in S:*
(i) *For every $a \in \mathbf{V}_{\leq}(S)$, $a \in K$.*
(ii) *For every $\mathcal{L}$-class of S that contains an element $a \in \mathbf{V}_{\leq}(S)$ is a subset of K.*

Proof. (i): Consider $a \in \mathbf{V}_{\leq}(S)$. Then for all $n \in \mathbb{N}$, $a \leq a(xa)^n$ for some $x \in S$. Since S is a nil-extension of K, there is some $m \in \mathbb{N}$ such that $(xa)^m \in K$. Now since K is an ideal of S, $a(xa)^m \in K$ and so $a \in K$.

(ii): Let L be an $\mathcal{L}$-class of S that contains an element $a \in \mathbf{V}_{\leq}(S)$. Now for some $y \in L$, there is some $s \in S$ such that $y \leq sa$. Then by (i), it follows that $y \in K$ and hence $L \subseteq K$. This completes the proof. ■

In the next theorem we describe ordered semigroups which are nil-extensions of both left simple and π-inverse ordered semigroups and show that S is a nil-extension of a left simple and π-inverse ordered semigroup if and only if S is a nil-extension of a t-simple and π-inverse ordered semigroup.

Theorem 2.4. *The following conditions on an ordered semigroup S are equivalent:*

(i) *S is a nil-extension of a left simple and π-inverse ordered semigroup;*
(ii) *S is π-inverse and l-Archimedean ordered semigroup;*
(iii) *S is π-inverse and $a\mathcal{L}^*b$ for every $a, b \in S$;*
(iv) *S is π-inverse and $e\mathcal{L}^*f$ for every $e, f \in E_{\leq}(S)$;*
(v) *S is π-regular and $e\mathcal{H}^*f$ for every $e, f \in E_{\leq}(S)$;*
(vi) *S is π-regular and $a\mathcal{H}^*b$ for every $a, b \in S$;*
(vii) *S is π-inverse and t-Archimedean ordered semigroup;*
(viii) *S is a nil-extension of t-simple and π-inverse ordered semigroup.*

Proof. (i)⇒(ii): Let S be a nil-extension of a left simple and π-inverse ordered semigroup K. Choose $a \in S$. Then there exists $k \in \mathbb{N}$ such that $a^k \in K$. Since K is π-inverse, for a^k there exists $r \in \mathbb{N}$ such that for any $x, y \in V_{\leq}(a^m) \subseteq K$ it gives $x\mathcal{H}y$, where $m = kr$. Hence S is a π-inverse ordered semigroup.

Now for every $b \in S$, as K is an ideal of S, we have $a^k b^n \in KS \subseteq K$ for every $n \in \mathbb{N}$. But K is left simple, so for $a^m, a^k b^n \in K$ there exists $z \in K$ such that $a^m \leq z a^k b^n$. Now $a^m \leq a^m x a^m \leq a^m x z a^k b^n$, that is $a^m \in (a^m S b^n]$ for every $n \in \mathbb{N}$. Hence S is l-Archimedean.

(ii)⇒(iii) and (iii)⇒(iv): These implications are obvious.

(iv)⇒(v): Since S is π-inverse, so S is π-regular and $e\mathcal{L}f$ implies $e\mathcal{H}f$ for any $e, f \in E_{\leq}(S)$, by Theorem 1.1. Hence $e\mathcal{H}^*f$.

(v)⇒(vi): Let $a, b \in S$. Since S is π-regular, we let $m, n \in \mathbb{N}$ be the smallest positive integers such that $a^m, b^n \in Reg_{\leq}(S)$. Then there exist $x, y \in S$ such that $a^m \leq a^m x a^m$ and $b^n \leq b^n y b^n$. Clearly $a^m x, x a^m, b^n y, y b^n \in E_{\leq}(S)$. Now $a^m \leq a^m x a^m \leq b^n y z a^m$ for some $z \in S$. So $a^m \leq b^n s_1$, where $s_1 = y z a^m$. Also $a^m \leq s_2 b^n$ for some $s_2 \in S$. Similarly there exists $s_3, s_4 \in S$ such that $b^n \leq s_3 a^m$ and $b^n \leq a^m s_4$. Hence $a\mathcal{H}^*b$.

(vi)⇒(vii): Let $a, b \in S$ and $a', a'' \in V_{\leq}(a^m)$ for some $m \in \mathbb{N}$. Hence from (vi), $a'a^m \mathcal{H}^* a''a^m$ and so $a'a^m \mathcal{H} a''a^m$. Now $a' \leq a'a^m a' \leq a''a^m t_1 a' = a'' t_2$, where $t_2 = a^m t_1 a'$. Similarly $a'' \leq a' t_3$ for some $t_3 \in S$. Hence $a'\mathcal{R}a''$. Similarly $a'\mathcal{L}a''$, thus $a'\mathcal{H}a''$. Also $a^m \in (b^n S b^n]$. Hence S is π-inverse and t-Archimedean.

(vii)⇒(viii): Suppose that S is π-inverse and t-Archimedean ordered semigroup. Since S is t-Archimedean, S is a nil-extension of a t-simple ordered semigroup K, by Corollary 1.3. So K is left simple. Let $a \in K$. Since S is π-inverse, for $a \in S$ there exists $m \in \mathbb{N}$ such that for every $a', a'' \in V_{\leq}(a^m) \subseteq S$ implies $a'\mathcal{H}a''$. Now as K is an ideal, $a'a^m a' \in K$. Hence $a' \leq a'a^m a'$ implies that $a' \in K$. Similarly $a'' \in K$. So $a'\mathcal{H}a''$ in K. Hence K is a π-inverse ordered semigroup.

(viii)$\Rightarrow$(i): This is obvious. ∎

Corollary 2.5. *The following conditions on an ordered semigroup S are equivalent:*

(i) *S is a nil-extension of a simple and π-inverse ordered semigroup;*
(ii) *S is π-inverse and $e\mathcal{J}^*f$ for all $e, f \in E_{\leq}(S)$;*
(iii) *S is π-inverse and $a\mathcal{J}^*b$ for all $a, b \in S$;*
(iv) *S is π-inverse and for all $a, b \in S$, there exists $m \in \mathbb{N}$ such that $a^m \in (SbS]$;*
(v) *S is π-inverse and Archimedean ordered semigroup.*

Proof. (i)$\Rightarrow$(ii): Let S be a nil-extension of a simple and π-inverse ordered semigroup K. Choose $a \in S$. Then there exists $k \in \mathbb{N}$ such that $a^k \in K$. Since K is π-inverse, for a^k there exists $r \in \mathbb{N}$ such that for any $x, y \in V_{\leq}(a^m) \subseteq K$ implies $x\mathcal{H}y$, where $m = kr$. Hence S is a π-inverse ordered semigroup.

Now for every $e, f \in E_{\leq}(S)$, as K is an ideal of S, we have $e, f \in K$. But K is simple, so for $e, f \in K$ there exists $u, v \in K$ such that $e \leq ufv$. Similarly $f \leq wez$ for some $w, z \in S$. So $e\mathcal{J}f$. Hence $e\mathcal{J}^*f$.

(ii)$\Rightarrow$(iii): Since S is π-inverse, S is π-regular. Let m, n be the smallest positive integers such that $a^m, b^n \in Reg_{\leq}(S)$. So there exist $x, y \in S$ such that $a^m \leq a^m x a^m$ and $b^n \leq b^n y b^n$. Clearly $a^m x, xa^m, b^n y, yb^n \in E_{\leq}(S)$. Now $a^m \leq a^m x a^m \leq s_1 y b^n s_2 a^m$ for some $s_1, s_2 \in S$. So $a^m \in (Sb^nS]$. Similarly $b^n \in (Sa^mS]$. Hence $a\mathcal{J}^*b$.

(iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(v): These implications are obvious.

(v)$\Rightarrow$(i): Clearly $Intra(S) \neq \phi$. Since S is an Archimedean ordered semigroup with $Intra(S) \neq \phi$, so S is a nil-extension of a simple ordered semigroup K, by Theorem 1.2. Let $a \in K$. Since S is π-inverse, for $a \in S$ there exists $m \in \mathbb{N}$ such that for any $a', a'' \in V_{\leq}(a^m) \subseteq S$ implies that $a'\mathcal{H}a''$. Now since K is an ideal, $a'a^m a' \in K$. Hence $a' \leq a'a^m a'$ implies $a' \in K$. Similarly $a'' \in K$. So $a'\mathcal{H}a''$ holds in K. Hence K is a π-inverse ordered semigroup. ∎

Theorem 2.6. *An ordered semigroup S is a nil-extension of an inverse ordered semigroup if and only if the following conditions hold in S:*

(i) *S is π-inverse;*
(ii) *for $a \in S$ and $b \in \mathbf{V}_{\leq}(S)$ such that $a \leq ba$ implies that $a \in \mathbf{V}_{\leq}(S)$;*
(iii) *for $a \in S$ and $b \in \mathbf{V}_{\leq}(S)$ such that $a \leq ab$ implies that $a \in \mathbf{V}_{\leq}(S)$;*
(iv) *for $a \in S$ and $b \in \mathbf{V}_{\leq}(S)$ such that $a \leq b$ implies that $a \in \mathbf{V}_{\leq}(S)$.*

Proof. First suppose that S is a nil-extension of an inverse ordered semigroup K.

(i). Let $a \in S$. Then there is $m \in \mathbb{N}$ such that $a^m \in K$. Since K is inverse, for every $x, y \in V_{\leq}(a^m) \subseteq K$ implies $x\mathcal{H}y$. Thus S is π-inverse.

(ii). Let $b \in \mathbf{V}_{\leq}(S)$ and $a \in S$ such that $a \leq ba$. Since $b \in Reg_{\leq}(S)$, there is $z \in S$ such that $b \leq b(zb)^n$ for all $n \in \mathbb{N}$. Let $n_1 \in \mathbb{N}$ be such that $(zb)^{n_1} \in K$. Then $b(zb)^{n_1} \in K$. This implies $b \in K$ and so $ba \in K$ and finally $a \in K$. Since K is an inverse ordered semigroup, $a \in \mathbf{V}_{\leq}(S)$.

(iii). This is similar to (ii).

(iv). Let $a \in S$ and $b \in \mathbf{V}_{\leq}(S)$ such that $a \leq b$. Clearly $b \in Reg_{\leq}(S)$, and so for some $z \in S$, $b \leq b(zb)^n$, for all $n \in \mathbb{N}$. Since S is a nil-extension of K, there is $m \in \mathbb{N}$ such that $(zb)^m \in K$ and so $b \in K$. Thus $a \in K$ and hence $a \in \mathbf{V}_{\leq}(S)$.

Conversely, assume that given conditions hold in S. Let $a \in S$ be arbitrary. Then by (i) there exists $m \in \mathbb{N}$ such that for any $x, y \in V_{\leq}(a^m) \subseteq S$ implies $x\mathcal{H}y$. So $\mathbf{V}_{\leq}(S) \neq \phi$. Say $T = \mathbf{V}_{\leq}(S)$. Thus for each $a \in S$, there exists $m \in \mathbb{N}$ such that $a^m \in T$. Now choose $s \in S$ and $a \in T$. Then $a \in Reg_{\leq}(S)$, so there is $h \in S$ such that $a \leq a(ha)^n$ for all $n \in \mathbb{N}$. Let $m_1 \in \mathbb{N}$ be such that $(ha)^{m_1} \in T$. So $sa \leq sa(ha)^{m_1}$ implies that $sa \in \mathbf{V}_{\leq}(S) = T$, by (iii). Similarly $as \in T$ follows from (ii).

Now consider $a \in S$ and $b \in T$ such that $a \leq b$. Then by (iv) $a \in T$. Hence T is an ideal. Also T is an inverse ordered semigroup. Hence S is a nil-extension of an inverse ordered semigroup T. ■

In the following results we characterize ordered semigroups which are complete semilattice of nil-extensions of different ordered semigroups.

Theorem 2.7. *Let S be an ordered semigroup. Then the following conditions are equivalent on S:*

(i) *S is a complete semilattice of nil-extensions of simple and π-inverse ordered semigroups;*

(ii) *S is a complete semilattice of nil-extensions of simple ordered semigroups and $\Pi Intra_{\leq}(S) = \Pi\mathbf{V}_{\leq}(S)$;*

(iii) *S is π-inverse and is a complete semilattice of Archimedean ordered semigroups.*

Proof. (i)⇒(ii): Let S be a complete semilattice of semigroups $\{S_\alpha\}_{\alpha \in Y}$ and ρ be the corresponding complete semilattice congruence on S. For $\alpha \in Y$, let S_α be a nil-extension of a simple and π-inverse ordered semigroup K_α. Here we need only to show that $\Pi Intra_{\leq}(S) = \Pi\mathbf{V}_{\leq}(S)$. For this, let $a \in \Pi\mathbf{V}_{\leq}(S)$. Then there are $m \in \mathbb{N}$ and $\alpha \in Y$ such that $a^m \in K_\alpha$. Now the simplicity of K_α yields that $a^m \in (K_\alpha a^{2m} K_\alpha]$. Thus $a \in \Pi Intra_{\leq}(S)$. Therefore $\Pi\mathbf{V}_{\leq}(S) \subseteq \Pi Intra_{\leq}(S)$.

Now let $b \in \Pi Intra_{\leq}(S)$. Then there are $x, y \in S$ and $\gamma \in Y$ such that $b^m \leq xb^{2m}y$ and $b \in S_\gamma$. Let S_γ be a nil-extension of a simple and π-inverse ordered semigroup K_γ. Now $b^m \leq (xb^m)^n b^m y^n$, for all $n \in \mathbb{N}$. Also by completeness of ρ we have that $(b^m)_\rho = (b^m x b^{2m} y)_\rho = (xb^m xb^m y)_\rho = (xb^m)_\rho (xb^m y)_\rho = (xb^m)_\rho (b^m)_\rho = (xb^m)_\rho$. Thus $xb^m \in S_\gamma$. So there is $m_1 \in \mathbb{N}$ such that $(xb^m)^{m_1} \in$

K_γ. Thus $(b^m)^{m_1} \in K_\gamma$. Since K_γ is π-inverse, it follows that $b \in \Pi\mathbf{V}_\leq(S)$. Therefore $\Pi Intra_\leq(S) \subseteq \Pi\mathbf{V}_\leq(S)$. Hence $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$.

(ii)⇒(iii): Suppose S is a complete semilattice of semigroups $S_\alpha(\alpha \in Y)$, where S_α is a nil-extension of a simple ordered semigroup K_α and $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$. Let $a \in S$. Then there are $m \in \mathbb{N}$ and $\alpha \in Y$ such that $a^m \in K_\alpha$. Since each K_α is simple, for $a^m \in K_\alpha$, $(a^m)^r \in (K_\alpha(a^m)^{2r}K_\alpha] \subseteq (S(a^m)^{2r}S]$, for all $r \in \mathbb{N}$. Thus $a \in \Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$. Hence S is a π-inverse ordered semigroup. Also S is a complete semilattice of Archimedean ordered semigroups, by Theorem 1.2.

(iii)⇒(i): Suppose that the condition (iii) holds. Then S is a complete semilattice of nil-extensions of simple ordered semigroups, by Theorem 1.2. Suppose that S is a complete semilattice of semigroups $S_\alpha(\alpha \in Y)$, where S_α is a nil-extension of a simple ordered semigroup K_α. Let $a \in K_\alpha$. Since S is π-inverse, there is $m \in \mathbb{N}$ such that for every $z, y \in V_\leq(a^m)$ in S implies that $z\mathcal{H}y$. By completeness of ρ, it gives $(a^m)_\rho = (za^m)_\rho$ and so $a^m, za^m \in S_\alpha$. Now $a^m \leq a^m z a^m$ implies that $a^m \leq a^m(za^mz)a^m$. Since S_α is a nil-extension of K_α, K_α is an ideal of S_α. Thus $za^mz \in K_\alpha$. Now by completeness of ρ, $z \leq za^mz$ gives $(z)_\rho = (za^mz)_\rho$. So $z \in K_\alpha$. Similarly $y \in K_\alpha$. Hence K_α is π-inverse and so S is a complete semilattice of nil-extensions of simple and π-inverse ordered semigroups. ∎

Corollary 2.8. *Let S be an ordered semigroup. Then the following conditions are equivalent on S:*

(i) *S is a complete semilattice of nil-extensions of left simple and π-inverse ordered semigroups;*

(ii) *S is a complete semilattice of nil-extensions of left simple ordered semigroups and $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$;*

(iii) *S is π-inverse and is a complete semilattice of l-Archimedean ordered semigroups.*

Proof. (i)⇒(ii): Let S be a complete semilattice of semigroups $\{S_\alpha\}_{\alpha\in Y}$ and ρ be the corresponding complete semilattice congruence on S. Let $\alpha \in Y$ and S_α be a nil-extension of a left simple and π-inverse ordered semigroup K_α. Here we need only to show that $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$. For this, let $a \in \Pi\mathbf{V}_\leq(S)$. Then there is $m \in \mathbb{N}$ such that $a^m \in K_\alpha$. Now the left simplicity of K_α yields that $a^m \in (a^{3m}K_\alpha]$, that is, $a^m \leq a^{3m}s_1 = a^m a^{2m} s_1$ for some $s_1 \in K_\alpha$. Thus $a \in \Pi Intra_\leq(S)$. Therefore $\Pi\mathbf{V}_\leq(S) \subseteq \Pi Intra_\leq(S)$.

Now let $b \in \Pi Intra_\leq(S)$. Then clearly $b \in \mathbf{V}_\leq(S)$. Therefore $\Pi Intra_\leq(S) \subseteq \Pi\mathbf{V}_\leq(S)$. Hence $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$.

(ii)⇒(iii): Suppose S is a complete semilattice of semigroups $S_\alpha(\alpha \in Y)$, where S_α is a nil-extension of a left simple ordered semigroup K_α and $\Pi Intra_\leq(S) = \Pi\mathbf{V}_\leq(S)$. Let $a \in S$. Then there are $m \in \mathbb{N}$ and $\alpha \in Y$ such that $a^m \in K_\alpha$. Since each K_α is left simple, for $a^m \in K_\alpha$ there exists $r \in \mathbb{N}$ such that $(a^m)^r \leq (a^m)^{3r}s_2 = a^m a^{2r} s_2$ for some $s_2 \in K_\alpha$. Thus

$a \in \Pi Intra_{\leq}(S) = \Pi \mathbf{V}_{\leq}(S)$. Hence S is a π-inverse ordered semigroup. Also S is a complete semilattice of l-Archimedean ordered semigroup by [2, Corollary 4.2]..

(iii)$\Rightarrow$(i): Suppose that the condition (iii) holds. Then S is a complete semilattice of nil-extensions of left simple ordered semigroups by [2, Corollary 4.2]. Suppose that S is a complete semilattice of semigroups $S_\alpha (\alpha \in Y)$, where S_α is a nil-extension of a left simple semigroup K_α. Clearly K_α is π-inverse, by Theorem 2.7 and so S is a complete semilattice of nil-extensions of left simple and π-inverse ordered semigroups. ∎

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