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Upper Bounds for the Numerical Radius of Hilbert Space Operators

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Abstract. We prove several norm and numerical radius inequalities for linear operators in Hilbert spaces. In particular, it is proved that if A is a bounded linear operator on a complex Hilbert space, then

$$\omega^2(A) \le \frac{1}{2}\left(\omega(|A^*|\,|A|) + \|A\|^2\right),$$

where $\omega(A)$, $\|A\|$, and $|A|$ are the numerical radius, the usual operator norm, and the absolute value of A, respectively.

Keywords: Numerical radius; Operator norm; Convex function; Positive operator.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $A \in \mathbb{B}(\mathbb{H})$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the operator norm of A, respectively. Recall that $\omega(A) = \sup_{x\in\mathbb{H},\|x\|=1} |\langle Ax, x\rangle|$ and $\|A\| = \sup_{x\in\mathbb{H},\|x\|=1} \|Ax\|$. It is well-known that if $A \in \mathbb{B}(\mathbb{H})$ and f is a non-negative increasing function on $[0, \infty)$, then $\|f(|A|)\| = f(\|A\|)$. Here $|A|$ stands for the positive operator $(A^*A)^{\frac{1}{2}}$.

It is easy to check that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1}$$

The inequalities in (1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal.

The numerical radius and the usual operator norm satisfy the following well-known inequalities

$$\left\|A^2\right\| \leq \|A\|^2 \text{ and } \omega\left(A^2\right) \leq \omega^2(A).$$

In [7], Kittaneh improved the second inequality in (1), and obtained the following result:

$$\omega(A) \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right). \tag{2}$$

He also showed the following estimate, which is stronger than (2),

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\|. \tag{3}$$

Another refinement of the second inequality in (1) has been established in [10]. This refinement asserts that if $A \in \mathbb{B}(\mathbb{H})$, then

$$\omega^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{4}$$

Also, in the same paper, the author proved that

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \leq \omega^2(A). \tag{5}$$

It can be easily seen that (5) improves the first inequality in (1). It should be mentioned here that upper bounds obtained in (2) and (4) are not comparable.

Recently, many mathematicians have obtained different numerical radius inequalities of Hilbert space operators, the interested readers are invited to see [5, 11, 12, 13, 14, 15, 16] and references therein. Here, we obtain several new inequalities for the numerical radius of the Hilbert space operators. The bounds obtained here improve on the existing bounds.

2. Numerical Radius Inequalities

Lemma 2.1. [2] *Let $x, y, z \in \mathbb{H}$. Then*

$$|\langle z, x\rangle| \, |\langle z, y\rangle| \leq \frac{\|z\|^2}{2} \left(|\langle x, y\rangle| + \|x\| \, \|y\|\right).$$

Our first result is stated as follows.

Theorem 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega^2 (A) \leq \frac{1}{2} \left(\left(\int_0^1 \left\| (1-v) |A|^2 + v|A^*|^2 \right\|^{\frac{1}{2}} \right)^2 + \omega \left(A^2\right) \right). \tag{6}$$

Proof. By using Lemma 2.1, the logarithmic-mean, and the Young inequality, we have

$$\begin{aligned}
& |\langle Ax, x\rangle|^2 \\
\leq & \frac{1}{2} \left(\|Ax\| \, \|A^* x\| + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
\leq & \frac{1}{2} \left(\left(\int_0^1 \|Ax\|^{1-v} \|A^* x\|^v dv \right)^2 + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
= & \frac{1}{2} \left(\left(\int_0^1 \left\langle |A|^2 x, x \right\rangle^{\frac{1-v}{2}} \left\langle |A^*|^2 x, x \right\rangle^{\frac{v}{2}} dv \right)^2 + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
= & \frac{1}{2} \left(\left(\int_0^1 \left(\left\langle |A|^2 x, x \right\rangle^{1-v} \left\langle |A^*|^2 x, x \right\rangle^{v} \right)^{\frac{1}{2}} dv \right)^2 + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
\leq & \frac{1}{2} \left(\left(\int_0^1 \left((1-v) \left\langle |A|^2 x, x \right\rangle + v \left\langle |A^*|^2 x, x \right\rangle \right)^{\frac{1}{2}} dv \right)^2 + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
= & \frac{1}{2} \left(\left(\int_0^1 \left\langle \left((1-v) |A|^2 + v|A^*|^2 \right) x, x \right\rangle^{\frac{1}{2}} dv \right)^2 + \left|\left\langle A^2 x, x\right\rangle\right| \right) \\
\leq & \frac{1}{2} \left(\left(\int_0^1 \left\| (1-v) |A|^2 + v|A^*|^2 \right\|^{\frac{1}{2}} dv \right)^2 + \omega \left(A^2\right) \right).
\end{aligned}$$

This implies that

$$|\langle Ax,x\rangle|^2 \le \frac{1}{2}\left(\left(\int_0^1 \left\|(1-v)\,|A|^2+v|A^*|^2\right\|^{\frac{1}{2}}\right)^2+\omega\left(A^2\right)\right).$$

By taking the supremum over all unit vector $x\in\mathbb{H}$, we reach the desired result. ■

In [4], Dragomir proved the following inequality

$$\omega^2(A)\le\frac{1}{2}\left(\|A\|^2+\omega\left(A^2\right)\right). \tag{7}$$

By the triangle inequality for the usual operator norm,

$$\left(\int_0^1 \left\|(1-v)\,|A|^2+v|A^*|^2\right\|^{\frac{1}{2}}\right)^2\le\|A\|^2,$$

we get

$$\begin{aligned}\omega^2(A)&\le\frac{1}{2}\left(\left(\int_0^1 \left\|(1-v)\,|A|^2+v|A^*|^2\right\|^{\frac{1}{2}}\right)^2+\omega\left(A^2\right)\right)\\&\le\frac{1}{2}\left(\|A\|^2+\omega\left(A^2\right)\right)\end{aligned}$$

and imply that our inequality (6) is an improvement of the Dragomir's inequality (7).

The following lemmas are also useful in the sequel.

Lemma 2.3. [3, (2.26)] *Let $x,y,z\in\mathbb{H}$. Then*

$$|\langle z,x\rangle|^2+|\langle z,y\rangle|^2\le\|z\|^2\left(|\langle x,y\rangle|+\max\left(\|x\|^2,\|y\|^2\right)\right).$$

Lemma 2.4. [9] *Let $A\in\mathbb{B}(\mathbb{H})$ and let $x,y\in\mathbb{H}$ be any vector. If f,g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)\,g(t)=t,(t\ge0)$, then*

$$|\langle Ax,y\rangle|\le\|f(|A|)\,x\|\,\|g(|A^*|)\,y\|.$$

In particular,

$$|\langle Ax,y\rangle|\le\sqrt{\left\langle|A|^{2(1-v)}x,x\right\rangle\left\langle|A^*|^{2v}y,y\right\rangle},\ \ (0\le v\le1).$$

A different upper bound for the numerical radius is incorporated in the following theorem.

Theorem 2.5. *Let* $A \in \mathbb{B}(\mathbb{H})$ *and let* $0 \leq t \leq 1$*. Then*

$$\omega^2(A) \leq \frac{1}{2}\left(\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \max\left(\|A\|^{4t}, \|A\|^{4(1-t)}\right)\right).$$

Proof. Let $0 \leq t \leq 1$. Put $x = |A|^{2t}x$, $y = |A^*|^{2(1-t)}x$, and $z = x$ with $\|x\| = 1$, in Lemma 2.3. Then

$$\begin{aligned} &\left\langle |A|^{2t}x, x\right\rangle^2 + \left\langle |A^*|^{2(1-t)}x, x\right\rangle^2 \\ &\leq \left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \max\left(\left\||A|^{2t}x\right\|^2, \left\||A^*|^{2(1-t)}x\right\|^2\right)\right). \end{aligned}$$

On the other hand, by Lemma 2.4,

$$2|\langle Ax, x\rangle|^2 \leq 2\left\langle |A|^{2t}x, x\right\rangle\left\langle |A^*|^{2(1-t)}x, x\right\rangle \leq \left\langle |A|^{2t}x, x\right\rangle^2 + \left\langle |A^*|^{2(1-t)}x, x\right\rangle^2.$$

Thus,

$$\begin{aligned} |\langle Ax, x\rangle|^2 &\leq \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \max\left(\left\||A|^{2t}x\right\|^2, \left\||A^*|^{2(1-t)}x\right\|^2\right)\right) \\ &= \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \max\left(\left\langle |A|^{4t}x, x\right\rangle, \left\langle |A^*|^{4(1-t)}x, x\right\rangle\right)\right) \\ &\leq \frac{1}{2}\left(\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \max\left(\|A\|^{4t}, \|A\|^{4(1-t)}\right)\right). \end{aligned}$$

This implies that

$$\omega^2(A) \leq \frac{1}{2}\left(\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \max\left(\|A\|^{4t}, \|A\|^{4(1-t)}\right)\right). \qquad \blacksquare$$

The following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let* $A \in \mathbb{B}(\mathbb{H})$*. Then*

$$\begin{aligned} \omega(A) &\leq \sqrt{\frac{1}{2}\left(\omega(|A^*|\,|A|) + \|A\|^2\right)} \\ &\leq \sqrt{\frac{1}{2}\left(\||A^*|\,|A|\| + \|A\|^2\right)} \\ &= \sqrt{\frac{1}{2}\left(\|A^2\| + \|A\|^2\right)} \\ &\leq \|A\|. \end{aligned}$$

Our last result is stated as follows.

Theorem 2.7. *Let* $A \in \mathbb{B}(\mathbb{H})$ *and let* $0 \le t \le 1$. *Then*

$$\omega^2(A) \le \frac{1}{2}\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \frac{1}{4}\left\||A|^{4t} + |A^*|^{4(1-t)}\right\|.$$

Proof. Let $0 \le t \le 1$. Put $x = |A|^{2t}x$, $y = |A^*|^{2(1-t)}x$, and $z = x$ with $\|x\| = 1$, in Lemma 2.3. Then

$$\begin{aligned}&\left\langle |A|^{2t}x, x\right\rangle \left\langle |A^*|^{2(1-t)}x, x\right\rangle \\ &\le \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \left\||A|^{2t}x\right\| \left\||A^*|^{2(1-t)}x\right\|\right).\end{aligned}$$

Hence, by Lemma 2.4, we deduce that

$$|\langle Ax, x\rangle|^2 \le \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \left\||A|^{2t}x\right\| \left\||A^*|^{2(1-t)}x\right\|\right).$$

So,

$$\begin{aligned}|\langle Ax, x\rangle|^2 &\le \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \left\||A|^{2t}x\right\| \left\||A^*|^{2(1-t)}x\right\|\right) \\ &= \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \sqrt{\left\langle |A|^{4t}x, x\right\rangle \left\langle |A^*|^{4(1-t)}x, x\right\rangle}\right) \\ &\le \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \frac{1}{2}\left(\left\langle |A|^{4t}x, x\right\rangle + \left\langle |A^*|^{4(1-t)}x, x\right\rangle\right)\right) \\ &= \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-t)}|A|^{2t}x, x\right\rangle\right| + \frac{1}{2}\left\langle \left(|A|^{4t} + |A^*|^{4(1-t)}\right)x, x\right\rangle\right) \\ &\le \frac{1}{2}\left(\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \frac{1}{2}\left\||A|^{4t} + |A^*|^{4(1-t)}\right\|\right),\end{aligned}$$

which implies,

$$\omega^2(A) \le \frac{1}{2}\omega\left(|A^*|^{2(1-t)}|A|^{2t}\right) + \frac{1}{4}\left\||A|^{4t} + |A^*|^{4(1-t)}\right\|. \qquad \blacksquare$$

By taking $t = 1/2$ in Theorem 2.7 we get a recent result proved by Heydarbeygi et al. in [6, Corollary 3.3].

3. Norm Inequalities

For any $a, b \in \mathbb{R}$, we have

$$\left(\frac{a+b}{2}\right)^2 = \frac{1}{2}\left(\frac{a^2+b^2}{2} + ab\right).$$

If we assume g is non-negative nondecreasing convex function on $[0,\infty)$, then by applying the arithmetic-geometric mean inequality, we can write

$$\begin{aligned} g\left(\left(\frac{a+b}{2}\right)^2\right) &= g\left(\frac{\frac{a^2+b^2}{2}+ab}{2}\right) \\ &\le \frac{1}{2}\left(g\left(\frac{a^2+b^2}{2}\right)+g\left(ab\right)\right) \\ &\le g\left(\frac{a^2+b^2}{2}\right) \\ &\le \frac{g\left(a^2\right)+g\left(b^2\right)}{2}. \end{aligned}$$

Defining $f\left(t\right)=g\left(\sqrt{t}\right)$, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left(f\left(\sqrt{\frac{a^2+b^2}{2}}\right)+f\left(\sqrt{ab}\right)\right) \le \frac{f\left(a\right)+f\left(b\right)}{2}. \tag{8}$$

A straightforward extension of the inequality (8) to positive operators is given in Theorem 3.3.

To achieve our goal, we need the following lemmas.

Lemma 3.1. [1, Corollary 2.6] *Let $A, B \in \mathbb{B}\left(\mathbb{H}\right)$ be two positive operators and let f be a non-negative nondecreasing convex function on $[0,\infty)$. Then*

$$\left\| f\left(\frac{A+B}{2}\right)\right\| \le \frac{1}{2}\left\| f\left(A\right)+f\left(B\right)\right\|.$$

Lemma 3.2. [8, Corollary 1] *Let $A, B \in \mathbb{B}\left(\mathbb{H}\right)$ be two positive operators. Then*

$$\left\| S-T\right\| \le \left\| S+T\right\|.$$

Theorem 3.3. *Let $A, B \in \mathbb{B}\left(\mathbb{H}\right)$ be two positive operators and let f be a non-negative and non-decreasing function on $[0,\infty)$ such that $g\left(t\right)=f\left(\sqrt{t}\right)$ is convex. Then*

$$\begin{aligned} \left\| f\left(\frac{A+B}{2}\right)\right\| &\le \frac{1}{2}\left\| f\left(\left(\frac{A^2+B^2}{2}\right)^{\frac{1}{2}}\right)+f\left(\left(\frac{AB+BA}{2}\right)^{\frac{1}{2}}\right)\right\| \\ &\le \frac{1}{2}\left\| f\left(A\right)+f\left(B\right)\right\|. \end{aligned} \tag{9}$$

In particular, it gives the following important result:

$$\left\|\left(\frac{A+B}{2}\right)^r\right\| \le \frac{1}{2}\left\|\left(\frac{A^2+B^2}{2}\right)^{\frac{r}{2}}+\left(\frac{AB+BA}{2}\right)^{\frac{r}{2}}\right\| \le \frac{1}{2}\left\| A^r+B^r\right\|, \tag{10}$$

for any $r \geq 2$.

Proof. It is easy to see that for positive operators A and B in $\mathbb{B}(\mathbb{H})$,

$$\left(\frac{A+B}{2}\right)^2 = \frac{1}{2}\left[\frac{A^2+B^2}{2} + \frac{AB+BA}{2}\right].$$

Since g is non-negative and convex on $[0,\infty)$, it follows that

$$\begin{aligned}
\left\| g\left(\left(\frac{A+B}{2}\right)^2\right)\right\| &= \left\| g\left(\frac{1}{2}\left[\frac{A^2+B^2}{2} + \frac{AB+BA}{2}\right]\right)\right\| \\
&\leq \frac{1}{2}\left\| g\left(\frac{A^2+B^2}{2}\right) + g\left(\frac{AB+BA}{2}\right)\right\| && (11)\\
&\leq \frac{1}{2}\left\| g\left(\frac{A^2+B^2}{2}\right)\right\| + \frac{1}{2}\left\| g\left(\frac{AB+BA}{2}\right)\right\| && (12)\\
&\leq \frac{1}{4}\left\| g\left(A^2\right) + g\left(B^2\right)\right\| + \frac{1}{2}\left\| g\left(\frac{AB+BA}{2}\right)\right\| && (13)\\
&= \frac{1}{4}\left\| g\left(A^2\right) + g\left(B^2\right)\right\| + \frac{1}{2} g\left(\left\|\frac{AB+BA}{2}\right\|\right) && (14)\\
&\leq \frac{1}{4}\left\| g\left(A^2\right) + g\left(B^2\right)\right\| + \frac{1}{2} g\left(\left\|\frac{A^2+B^2}{2}\right\|\right) && (15)\\
&= \frac{1}{4}\left\| g\left(A^2\right) + g\left(B^2\right)\right\| + \frac{1}{2}\left\| g\left(\frac{A^2+B^2}{2}\right)\right\| && (16)\\
&\leq \frac{1}{2}\left\| g\left(A^2\right) + g\left(B^2\right)\right\|, && (17)
\end{aligned}$$

where the inequalities (11), (13), and (17) follows from Lemma 3.1, the inequality (12) obtained from the triangle inequality for the usual operator norm, and (15) is an immediate consequence of Lemma 3.2. Indeed,

$$\begin{aligned}
\|AB+BA\| &= \frac{1}{2}\left\|(A+B)^2 - (A-B)^2\right\| \\
&\leq \frac{1}{2}\left\|(A+B)^2 + (A-B)^2\right\| \\
&= \left\|A^2+B^2\right\|.
\end{aligned}$$

So, we have shown that

$$\begin{aligned}
&\left\| g\left(\left(\frac{A+B}{2}\right)^2\right)\right\| \\
\leq &\frac{1}{2}\left\| g\left(\frac{A^2+B^2}{2}\right) + g\left(\frac{AB+BA}{2}\right)\right\| \leq \frac{1}{2}\left\| g\left(A^2\right) + g\left(B^2\right)\right\|.
\end{aligned}$$

Now, since $f(t) = g\left(\sqrt{t}\right)$, we conclude (9).

Specializing the inequality (9) to the function $f(t) = t^r$ for $r \geq 2$, we obtain (10). This completes the proof of the theorem. ■

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