Southeast Asian Bulletin of Mathematics (2024) 48: 247–258

Southeast Asian
Bulletin of
Mathematics

L^r-Henstock-Kurzweil Integral on Finite Dimensional Banach Spaces

Hemanta Kalita
Department of Mathematics, Assam Don Bosco University, Sonapur, Guwahati, Assam 782402, India
Email: hemanta30kalita@gmail.com; hemanta.kalita@dbuniversity.ac.in

Tomás Pérez Becerra
Instituto de física y matemáticas Universidad Tecnológica de la Mixteca Carretera a Acatlima Km 2.5, Huajuapan de León, Oaxaca, Mexico. C.P. 69000
Email: tomas@mixteco.utm.mx

Hemen Bharali
Department of Mathematics, Assam Don Bosco University, Sonapur, Guwahati, Assam 782402, India
Email: hemen.bharali@dbuniversity.ac.in

Received 27 February 2020
Accepted 15 August 2021

Communicated by Peichu Hu

AMS Mathematics Subject Classification(2020): 26A39, 26A42

Abstract. We introduce L^r-Henstock-Kurzweil integral for finite dimensional Banach spaces. We discuss its properties. In this study we discuss L^r-Henstock-Kurzweil integral generalized Henstock-Kurzweil integral for finite dimensional Banach spaces.

Keywords: L^r-Henstock-Kurzweil integral; Banach valued L^r-Henstock-Kurzweil integral.

1. Introduction

The Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integrals are the extension of Dunford, Pettis, and Bochner integrals respectively. These integrals were defined and studied by Gordon [6]. Gordon showed that a Denjoy-Dunford

(Denjoy-Bochner) integrable function on $[a, b]$ is Dunford (Bochner) integrable in some interval of $[a, b]$ and that for the spaces that do not contain copy c_0, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some sub interval of $[a, b]$. H. Yoon et al. in [21] defined and studied the AP-Henstock extension of Dunford, Pettis, and Bochner integrals of functions mapping $[a, b]$ into Banach space X, respectively. Major and minor functions were first introduced by de la Vallée Poussin in his study of the properties of the Lebesgue integral and those of functions additive of a set (see [16]). Entirely equivalent notions were introduced independently by O. Perron [14], based on them a new definition of integral, which does not require the theory of measure were discussed. Calderón and Zygmund first gave the notion of derivation in L^r. Unlike the idea of the approximate derivative, it had proven to be quite effective in applications of partial differential equation, area of surfaces, etc. (see [1]). L. Gordon defined the notion of Dini derivatives in metric L^r (briefly L^r-derivatives). Also in his work, he discussed Perron integral in L^r (see [5]). Gordon proved that AP-derivatives are equivalent to L^r- derivatives. P.M. Musial and Y. Sagher introduced the L^r-Henstock-Kurzweil integral in [11]. P. Musial and F. Tulone describe a norm on the space of HK_r -integrable functions, as well as the dual and completion of this space (see [13]). P. Musial define the class of L^r-variational integrable functions and he had shown that it is equivalent to the class of L^r- Henstock-Kurzweil integrable functions. They also defined the class of functions of L^r-bounded variation (see [12]). L.D. Piazza et al. in [15] shows that variational Henstock-Kurzweil integral is equivalent to Kuzweil-Henstock integral for Banach space valued functions. In this paper we define L^r- Henstock-Kurzweil integral of finite dimensional Banach space valued functions define in $[a, b]$.

2. Preliminaries

In this paper, X denotes a real Banach space and X^* its dual. $B(X^*) = \{x^* \in X^* : ||x^*|| \leq 1\}$ is the unit ball in X^*.

To make our presentation reasonably self-contained, we recall a few definitions and results in this section that will be used in our main section.

Definition 2.1. [9, Definition 2.1] *A function $f : [a, b] \to X$ is said to be Henstock integrable on $[a, b]$ if there exists $A \in X$ with the following property: given $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that*

$$\left|\left|\sum_{i=1}^{\mathcal{P}} f(\xi_i)|I_i| - A\right|\right| < \epsilon$$

for each $\delta-$fine $\mathcal{P}-$partition $\{(I_i, \xi_i)\}_{i=1}^{\mathcal{P}}$ of $[a, b]$. We write A as $H\int_{[a,b]} f$.

Recall the family of all compact sub intervals $J,\ L \subset I = [a, b]$, a function $F : I \to X$ is additive if $F(J \cup L) = F(J) + F(L)$ for any non overlapping

$J, L \in I$ such that $J \cup L \in I$.

Definition 2.2. [18, Definition 3.6.1] *A function $f : I = [a, b] \to X$ is said to be strongly Henstock-Kurzweil integrable on $I = [a, b]$ if there is an additive function $F : I = [a, b] \to X$ such that for every $\epsilon > 0$ there exists a gauge δ on $I = [a, b]$ such that*

$$\sum_{i=1}^{k} \left\| f(t_i)|J_i| - F(J_i) \right\|_X < \epsilon$$

for every δ-fine $\mathcal{P}$-partition $\{(t_i, J_i),\ i = 1, 2, ..., k\}$ of $I = [a, b]$.

Recall the space $L^r,\ 1 \le r < \infty$, as

$$L^r([a,b]) = \left\{ f : \left(\frac{1}{h} \int_a^b |f(x) - P(x)|^r dx \right)^{\frac{1}{r}} < \epsilon,\ 0 < h < \infty, \right.$$
$$\left. \textit{for some polynomial } P(x) \right\}.$$

More about $L^r\big([a, b]\big)$, one can follow [1, 11, 19].

Definition 2.3. [11] *Let $f \in L^r(I)$ where $1 \le r < \infty$ and $I = (a, b)$. For all $x \in I$, recalling the r- Dini derivatives. In all cases below $h \to 0^+$.*

The upper-right L^r- derivative:

$$D_r^+ f(x) = \inf \left\{ a : \left(\frac{1}{h} \int_0^h [f(x+t) - f(x) - at]_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}.$$

The lower-right L^r- derivate:

$$D_{+,r} f(x) = \sup \left\{ a : \left(\frac{1}{h} \int_0^h [f(x+t) - f(x) - at]_-^r dt \right)^{\frac{1}{r}} = o(h) \right\}.$$

The upper-left L^r- derivate:

$$D_r^- f(x) = \inf \left\{ a : \left(\frac{1}{h} \int_0^h [-f(x-t) + f(x) - at]_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}$$

and the lower-left L^r- derivate:

$$D_{-,r} f(x) = \sup \left\{ a : \left(\frac{1}{h} \int_0^h [-f(x-t) + f(x) - at]_-^r dt \right)^{\frac{1}{r}} = o(h) \right\}.$$

Let $1 \le r < \infty$. Then L^r-upper derivate of f at x, $\overline{D}_r f(x)$, is defined by

$$\overline{D}_r f(x) = \min \left\{ D_r^- f(x), D_r^+ f(x) \right\}.$$

The L^r- lower derivative of f at x, $\underline{D}_r f(x)$, is defined by

$$\underline{D}_r f(x) = \min\left\{D_{-,r}f(x),\ D_{+,r}f(x)\right\}.$$

If $\underline{D}_r f(x) = \overline{D}_r f(x)$ then f has an L^r-derivative and it is denoted by $f'_r(x)$.

Remark 2.4. $D^+_r f(x) = \inf\left\{a :\ \int_0^h \left(\frac{f(x+t)-f(x)}{t} - a\right)^r_+ dt = o(h)\right\}$, with similar results for the other r-Dini derivatives.

Definition 2.5. [11] *For $1 \leq r < \infty$, a real valued function f is L^r- Henstock-Kurzweil integrable (in short HK_r- integrable) if there exists a function $F \in L^r[a,b]$ so that for any $\epsilon > 0$ there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non overlapping tagged intervals in $[a,b]$ with*

$$\mathcal{P} < \delta, \tag{1}$$

we have:

$$\sum_{i=1}^{n}\left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r dy\right)^{\frac{1}{r}} < \epsilon. \tag{2}$$

If (1) implies (2), then δ is HK_r-appropriate for ϵ and f. F is an HK_r-integral of f. Recall that a gauge δ is HK_r-appropriate for ϵ and for f if (2) holds for any $\delta-$fine tagged partition $\mathcal{P}$.

Let χ_E be the characteristic function on E. Then the function f is said to be L^r-Henstock-Kurzweil integrable on the set $E \subset [a,b]$ if the function $f.\chi_E$ is L^r-Henstock-Kurzweil integrable on $[a,b]$. We write

$$(L^r - H)\int_I f.\chi_E = (L^r - H)\int_E f.$$

If f is HK_r-integrable on $[a,b]$, the following function is well defined for all $x \in [a,b]$:

$$F(x) = (HK_r)\int_a^x f(t)dt. \tag{3}$$

Let $f \in HK_r[a,b]$. The HK_r norm of f is defined as follows:

$$||f||_{HK_r} = ||F||_r,$$

where F is the indefinite HK_r integral of f as defined in (3).

Definition 2.6. [11, Definition 11] *Let $1 \leq r < \infty$. We say that $F \in AC_r(E)$ if for all $\epsilon > 0$ there exists $\nu > 0$ and a gauge function $\delta(x)$ defined on E so that for all $\mathcal{P} = \{(x_i, [c_i, d_i])\} < \delta_E$ such that $\sum_{i=1}^{q}(d_i - c_i) < \nu$ we have*

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < \epsilon.$$

Definition 2.7. [12] *Let $r \geq 1$, let $f : [a, b] \to \mathbb{R}$ and let E be a measurable subset of $[a, b]$. Then f is L^r- bounded variation on E ($f \in BV_r(E)$) if there exists $M > 0$ and a gauge $\delta > 0$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of δ- fine tagged sub-intervals of $[a, b]$ having tags in E, such that*

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M.$$

In Definition 2.7 P. Musial missed the coherent concept of $L^r[a, b]$. We rewrite the definition of L^r-bounded variatioin as follows:

Definition 2.8. *Let $r \geq 1$, let $f : [a, b] \to \mathbb{R}$ and let E be a measurable subset of $[a, b]$. We say that f is L^r- bounded variation on E ($f \in BV_r(E)$) if there exists a function $F \in L^r([a, b])$ so that for any $M > 0$ and a gauge $\delta > 0$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$ fine tagged sub-intervals of $[a, b]$ having tags in E, such that*

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M.$$

P. Musial in [12] mention the sketch of proof of the following theorem. We will complete the proof.

Theorem 2.9. [12, Theorem 2] *If $f \in BV_r(E)$ then we can find $\{E_i\}_{i \geq 1}$ so that $E = \bigcup_{i=1}^{\infty} E_i$ and $f \in BV(E_i)$ for all i.*

Proof. Let $f \in BV_r(E)$. Then for a function $F \in L^r([a, b])$ there exists $M > 0$ and a gauge $\delta > 0$ defined on E so that $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$fine tagged sub intervals of $[a, b]$ having tags in E. Then

$$\sum_{i=1}^{n} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M. \tag{4}$$

Assume that $F \in BV_r[a, b]$ and let $\epsilon > 0$. Then for a gauge function δ defined on $[a, b]$ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\} < \delta$ such that Eq. (4) holds.

The function F is L^r-continuous and so clearly approximately continuous. Using [11, Theorem 5], there exists $\mathcal{P}_i = \{(x_{i,j}, [c_{i,j}, d_{i,j}])\} < \delta$ where $[c_{i,j}, d_{i,j}] \subseteq [c_i, d_i]$ for all i and j, so that

$$\sum_{i=1}^{n} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})| dy \geq \tfrac{1}{2} |F(d_i) - F(c_i)|.$$

Since $\mathcal{P} = \bigcup_{i=1}^{n} P_i$ is sub-ordinates to δ, we have

$$\sum_{i=1}^{n} |F(d_i) - F(c_i)| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j}^{n} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})| dy < \frac{1}{2}\epsilon.$$

So, $F \in BV(E_i)$. Hence we can find $f \in BV(E_i)$. ■

3. L^r-Henstock-Kurzweil Integral for Finite Dimensional Banach Spaces

In this section we define L^r-Henstock-Kurzweil integral for functions mapping $[a, b]$ into a Banach space X. Throughout the section our Banach space is finite dimensional.

An L^r-neighbourhood (or r-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose a r-nbd $S_x \subset [a, b]$ of x. Then we say that $\nabla = \{S_x : x \in E\}$ is a choice on E. A tagged interval $(x, [c, d])$ is said to be sub-ordinate to the choice $\nabla = \{S_x\}$ if $c, d \in S_x$.

Definition 3.1. *Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite non overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is sub-ordinate to the choice ∇ for each i, then we say that $\mathcal{P}$ is sub-ordinate to the choice ∇. If $\mathcal{P}$ is sub-ordinate to ∇ and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$. We say that $\mathcal{P}$ is tagged partition of $[a, b]$ that is sub-ordinate to ∇.*

Let $E \subset [a, b]$. If $\mathcal{P}$ is sub-ordinate to ∇ and each $x_i \in E$, $\mathcal{P}$ is called E-sub-ordinate to ∇. For a tagged partition $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ of $[a, b]$. We notate $S(f, \mathcal{P}) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)$.

Since X is a finite dimensional Banach space, from the fundamental Stone-Weierstrass theorem every continuous function on the unit ball B_X can be uniformly approximated by polynomials. So, we can state our definition as follows:

Definition 3.2. *For $1 \leq r < \infty$, a function $f : [a, b] \to X$ is called L^r−Henstock-Kurzweil integrable if there exists a function $F \in L^r[a, b]$ for any $\epsilon > 0$ and there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i, [c_i, d_i])_{i=1}^{n}\}$ of non overlapping tagged integrals in $[a, b]$ with $\mathcal{P} < \delta$ such that*

$$\sum_{i=1}^{n} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} ||F(y) - F(x_i) - f(x_i)(y - x_i)||_X^r dy \right)^{\frac{1}{r}} < \epsilon.$$

That is, a function $f : [a, b] \to X$ is L^r-Henstock-Kurzweil integrable on $[a, b]$ if there exists a vector $A \in X$ with the following property:

For each $\epsilon > 0$ there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non over-lapping tagged intervals in $[a, b]$ with a choice $\nabla < \delta$ on $[a, b]$ such that $||S(f, \mathcal{P}) - A||_X < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is sub-ordinate to ∇. Here $A = F(y) - F(x_i)$ where $F \in L^r[a, b]$.

The vector A is called L^r-Henstock-Kurzweil integral of f on $[a, b]$ and we denote it by $(L^r - AH)\int_a^b f$. The function f is L^r-Henstock-Kurzweil integrable on a measurable subset E of $[a, b]$ if $f.\chi_E$ is L^r-Henstock integrable on $[a, b]$. The collection of all function that are L^r-Henstock-Kurzweil integrable on $f : I = [a, b] \to X$, will be denoted by $AH_r(I)$.

Remark 3.3. If X is infinite dimensional, then Stone-Weierstrass theorem will not support us. So we can not state the definition of L^r-Henstock-Kurzweil integral like Definition 3.2 for finite dimensional case.

To find Banach-valued L^r-Henstock-Kurzweil integral, we may have two possibilities:

(i) We need to construct X as Banach algebra or

(ii) We need to redefine $L^r[a, b]$ as:

$$L^r([a,b]) = \Big\{ f : \ \big(\frac{1}{h}\int_a^b |f(x) - P(x)|^r dx\big)^{\frac{1}{r}} < \epsilon, \ 0 < h < \infty, \ \textit{for some non zero vectors } P(x) \Big\},$$

for $1 \le r < \infty$, where the dimension of $P(x) < h$.

Next we investigate in detailed about this.

4. Simple Properties

In this section we discuss a few basic properies of L^r-Henstock-Kurzweil integrable functions for finite dimensional Banach spaces.

Theorem 4.1. *If $f : I \to X$ is L^r-Henstock-Kurzweil integrable on I, then f is L^r-Henstock-Kurzweil integrable on each subinterval I_0 of I.*

Theorem 4.2.

(i) *If $f, g \in AH_r(I)$ then $f + g \in AH_r(I)$ and*

$$(L^r - AH)\int_I (f + g) = (L^r - AH)\int_I f + (L^r - AH)\int_I g.$$

(ii) *If* $f \in AH_r(I)$ *and* $k \in \mathbb{R}$ *then* $kf \in AH_r(I)$ *and* $(L^r - AH)\int_I kf = k(L^r - AH)\int_I f$.

The following simple convergence theorem for the L^r-Henstock-Kurzweil integral is rather important to note.

Theorem 4.3. *Let* $f_n : [a,b] \to X$ *be a* L^r*-Henstock-Kurzweil integrable function on* $[a,b]$*, for each positive integer* n *and suppose that* $f_n \to f$ *pointwise on* $[a,b]$*. If the sequence* $\{f_n\}$ *is uniformly* L^r*-Henstock-Kurzweil integrable on* $[a,b]$ *then* f *is* L^r*-Henstock-Kurzweil integrable on* $[a,b]$ *and* $\int_a^b f_n \to \int_a^b f$.

For each $\epsilon > 0$ there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non over-lapping tagged intervals in $[a,b]$ with a choice $\nabla < \delta$ on $[a,b]$ such that $||S(f,\mathcal{P}) - A||_r < \phi(d_i) - \phi(c_i)$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is sub-ordinate to ∇ with a non-decreasing function $\phi : [a,b] \to \mathbb{R}$ and a gauge $\delta > 0$ so that $\phi(b) - \phi(a) < \epsilon$.

Theorem 4.4. *Let* $f : [a,b] \to X$ *be* L^r*-Henstock-Kurzweil integrable on* $[a,b]$*. Then the following statements hold:*

(i) *For each* $x^* \in X^*$ *the function* $x^* f$ *is* L^r*-Henstock-Kurzweil integrable on* $[a,b]$ *and* $(L^r - AH)\int_a^b x^* f = x^*(L^r - AH)\int_a^b f$.

(ii) *If* $T : X \to Y$ *is continuous linear operator then* $(L^r - AH)\int_a^b Tf = T(L^r - AH)\int_a^b f$.

Proof. (i) Let $x^* \in X^*$. Since $f : [a,b] \to X$ is L^r-Henstock-Kurzweil integrable on $[a,b]$, for each $\nu > 0$ there exists a choice ∇ on $[a,b]$ such that

$$||S(f,\mathcal{P}) - (L^r - AH)\int_a^b f||_X < \frac{\nu}{||x^*||},$$

whenvere $\mathcal{P}$ is a tagged partition of $[a,b]$ that is sub-ordinate to ∇. Then,

$$\begin{aligned}||S(x^* f,\mathcal{P}) - x^*(L^r - AH)\int_a^b f||_X &\leq ||x^*||||S(f,\mathcal{P}) - (L^r - AH)\int_a^b f||_X \\ &< \nu.\end{aligned}$$

Therefore $x^* f$ is L^r-Henstock-Kurzweil integrable on $[a,b]$ and $(L^r - AH)\int_a^b x^* f = x^*(L^r - AH)\int_a^b f$.

(ii) We can use the similar technique as we have used in (i). ■

Theorem 4.5. *If* $f = 0_X$ *(the zero of* X*) a.e. on* $[a,b]$ *then* f *is* L^r*-Henstock-Kurzweil integrable on* $[a,b]$ *and* $(L^r - AH)\int_a^b f = 0_X$.

Proof. Since $||f||_X = 0$ a.e. on $[a,b]$, $||f||_X$ is L^r-Henstock-Kurzweil integrable on $[a,b]$. For any $\epsilon > 0$ there is a choice ∇ on $[a,b]$ such that $||f||_X(\mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is sub-ordinate to ∇. Let $\mathcal{P}$ be a tagged partition on $[a,b]$ that is sub-ordinate to ∇. Then

$$||f(\mathcal{P}) - 0_X||_X = ||f(\mathcal{P})||_X \leq ||f||_X(\mathcal{P}) < \epsilon.$$

Therefore f is L^r-Henstock integrable on $[a,b]$ and $(L^r - AH)\int_a^b f = 0_X$. ■

Remark 4.6. Let $f : [a,b] \to X$ be L^r-Henstock-Kurzweil integrable on $[a,b]$. If $f = g$ a.e. on $[a,b]$ then g is L^r-Henstock-Kurzweil integrable on $[a,b]$ and $(L^r - AH)\int_a^b f = (L^r - AH)\int_a^b g$.

Theorem 4.7. *Let $f : I = [a,b] \to X$ be $(L^r - AH)$-Henstock-Kurzweil integrable on $I = [a,b]$. Then f is weakly measurable.*

Proof. The proof is direct consequence of Theorem 4.5 and Remark 4.6. ■

Definition 4.8.

(i) *A function $f : [a,b] \to X$ is said to be scalarly L^r-Henstock-Kurzweil integrable on $[a,b]$ if for each x^* in X^* the function x^*f is L^r-Henstock-Kurzweil integrable on $[a,b]$.*

(ii) *A family $B \subset (L^r - AH)\big([a,b], X\big)$ is L^r-Henstock-Kurzweil equi-integrable on $[a,b]$ if for each $\epsilon > 0$, there exists a gauge function δ so that for all finite collections $\mathcal{P} = \big\{(x_i, [c_i, d_i])\big\}$ of non over-lapping tagged intervals in $[a,b]$ with a choice $\nabla < \delta$ on $[a,b]$ such that*

$$\sup_{f \in B} ||S\big(f, \mathcal{P}\big) - (L^r - AH)\int_a^b f||_X < \epsilon.$$

Proposition 4.9. *If the function $f : [a,b] \to X$ is scalarly L^r-Henstock-Kurzweil integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\bigcup_{k=1}^{\infty} X_k = [a,b]$. Then f is Dunford integrable on each X_k and*

$$\lim_{k\to\infty} (Dunford)\int_{X_k \cap [a,x]} f = (L^r - AH)\int_a^x f \ (scalarly).$$

Proof. The function $f : [a,b] \to X$ is scalarly L^r-Henstock-Kurzweil integrable on $[a,b]$ if x^*f is L^r-Henstock-Kurzweil integrable on $[a,b]$. Now using [10, Lemma 1] in the similar way, if x^*f is L^r-Henstock-Kurzweil integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\bigcup_{k=1}^{\infty} X_k = [a,b]$, x^*f is Lebesgue integrable on each X_k and

$$\lim_{k\to\infty} (L)\int_{X_k \cap [a,x]} x^*f = (L^r - AH)\int_a^x x^*f$$

uniformly on $[a, b]$ for each $x^* \in X^*$. This gives f is Dunford integrable on each X_k and

$$\lim_{k\to\infty} (Dunford) \int_{X_k \cap [a,x]} f = (L^r - AH) \int_a^x f \ (scalarly). \qquad \blacksquare$$

Theorem 4.10. *Suppose that X contains no copy of c_0 and $f : [a, b] \to X$. If the function f is L^r- scalarly Henstock-Kurzweil integrable on $[a, b]$, then each perfect set in $[a, b]$ contains a portion on which f is Pettis integrable.*

Proof. Since the function $f : [a, b] \to X$ is L^r- scalarly Henstock-Kurzweil integrable on $[a, b]$, by the definition of scalar Henstock-Kurzweil integral, x^*f is L^r-Henstock-Kurzweil integrable on $[a, b]$. Then Theorem 4.7 implies that x^*f is weakly measurable on $[a, b]$, and so f is Dunford integrable on $[a, b]$. Using [3, Theorem 7], f is Pettis integrable on $[a, b]$. ∎

Theorem 4.11. *Suppose that X contains no copy of c_0 and $f : [a, b] \to X$ is a measurable. If the function $f : [a, b] \to X$ is scalarly L^r-Henstock-Kurzweil integrable on $[a, b]$, then there exists a sequence $\{X_k\}$ of closed sets with $X_k \uparrow [a, b]$ such that for each $x^* \in X^*$, f is Pettis integrable on each X_k and*

$$\lim_{k\to\infty} (Pettis) \int_{X_k} f = (L^r - AH) \int_a^b f \ (scalarly).$$

Proof. Since f is scalarly L^r-Henstock-Kurzweil integrable on $[a, b]$, from Proposition 4.7, f is Dunford integrable on $[a, b]$. Remaining part of the proof is analogous to that of Theorem 4.10. ∎

Corollary 4.12. *Suppose that X contains no copy of c_0. If the function $f : [a, b] \to X$ is weakly L^r-Henstock-Kurzweil integrable on $[a, b]$, then there exists a sequence $\{X_k\}$ of closed sets, $\bigcup_{k=1}^{\infty} X_k = [a, b]$, f is Pettis integrable on each X_k.*

Using the notion of L^r-Henstock-Kurzweil equi-integrability, we may characterize the vector valued L^r-Henstock-Kurzweil integrable functions.

Theorem 4.13. *A function $f : [a, b] \to X$ is L^r-Henstock-Kurzweil integrable on $[a, b]$ if and only if the family $\{x^*f : \ x^* \in B(X^*)\}$ is L^r-Henstock-Kurzweil equi-integrable on $[a, b]$.*

Proof. Let $f : [a, b] \to X$ be L^r-Henstock-Kurzweil integrable on $[a, b]$. Then for each $\epsilon > 0$ there exists a choice $\nabla < \delta$ on $[a, b]$ such that:

$$\left|\left| S(f, \mathcal{P}) - (L^r - AH) \int_a^b f \right|\right|_r < \epsilon,$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is sub-ordinate to ∇.

Now,

$$\left|\left|S(f,\mathcal{P}) - (L^r - AH)\int_a^b f\right|\right|_r = \sup_{x^*\in B(X^*)}\left|S(x^*f,\mathcal{P}) - x^*(L^r - AH)\int_a^b f\right|.$$

Therefore $\{x^*f\ :\ x^* \in B(X^*)\}$ is L^r-Henstock-Kurzweil equi-integrable on $[a,b]$.

Conversely, suppose the family $\{x^*f\ :\ x^* \in B(X^*)\}$ is L^r-Henstock-Kurzweil equi-integrable on $[a,b]$. We need to show that $f : [a,b] \to X$ is L^r-Henstock-Kurzweil integrable on $[a,b]$. As $\{x^*f\ :\ x^* \in B(X^*)\}$ is L^r-Henstock-Kurzweil equi-integrable on $[a,b]$ then for each $\epsilon > 0$, there exists a gauge function δ so that for all finite collections $\mathcal{P} = \{(x_i,[c_i,d_i])\}$ of non-overlapping tagged intervals in $[a,b]$ with a choice $\nabla < \delta$ on $[a,b]$ such that

$$\sup_{x^*\in B(X^*)}\left|\left|S(x^*f,P) - (L^r - AH)\int_a^b x^*f\right|\right|_r < \epsilon. \tag{5}$$

By the definition of equi-integrability, Eq. (5) implies

$$\sup_{||x^*||\leq 1}\left|(L^r - AH)\int_a^b x^*f - S(x^*f,P)\right| < \epsilon, \tag{6}$$

where $\mathcal{P}$ is a tagged partition of $[a,b]$ which is sub-ordinate to ∇.

Define $T_f : X^* \to \mathbb{R}$ by $T_f(x^*) = (L^r - AH)\int_a^b x^*f$, also for each $a \in \mathbb{R}$ assume $Q(a) = \{x^* \in X^*\ :\ T_f(x^*) \leq a\}$. As $Q(a)$ is convex, by Banach-Dieudonne Theorem $Q(a) \cap B(X^*)$ is w^*-closed. Let x_0^* be a w^*-closure point of $Q(a) \cap B(X^*)$ and let $(x_r^*)_{r\in I} \subset Q(a) \cap B(X^*)$ be a net converging to x_0^* in the w^*-topology. Let the tagged partition $\mathcal{P}$ of $[a,b]$ be defined as $\mathcal{P} = \{(I_1,t_1),(I_2,t_2),...,(I_p,t_p)\}$ which is sub-ordinate to ∇. Now the convergence of $(x_r^*)_{r\in I}$, for $r_0 \in I$ gives,

$$\sum_{i=1}^{p}\left|x_{r_0}^*f(t_i) - x_0^*f(t_i)\right| < \epsilon. \tag{7}$$

Since $x_0^* \in B(X^*)$, from Eqs. (6) and (7), we can find $T_f(x_0^*) < a + \epsilon$. Therefore T_f is w^*-continuous and as X is the w^*-dual of X^*, and there exists $\mathcal{N} \in X$ such that $x^*(\mathcal{N}) = T_f(x^*)$. That is $T_f : X^* \to \mathbb{R}$ such that $T_f(x^*) = (L^r-AH)\int_a^b x^*f$. ■

We see that the L^r-Henstock-Kurzweil integral generalised the Henstock-Kurzweil integral for finite dimensional Banach spaces.

Theorem 4.14. *Let* $1 \leq r < \infty,\ f \in HK[a,b]$ *and* $F(x) = (HK)\int_a^x f$. *Then*

$$f \in AH_r[a,b],$$
$$F(x) = (L^r - AH)\int_a^x f.$$

Proof. The proof is analogous to that of [11, Theorem 9]. ■

Acknowledgement. The authors would like to thank the reviewer for reading the manuscript carefully and making valuable suggestions that significantly improve the presentation of the paper.

References

[1] A.P. Calderón and A. Zygmund, Local properties of solutions of elliptic partial differential equations, *Studia. Mathematica* **20** (1961) 171–225.

[2] S. Cao, The Henstock integral for Banach-valued functions, *Southeast Asian Bull. Math.* **16** (1992) 35–40.

[3] J. Diestel, J. Joseph, Uhl, Jr, *Vector Measures, Math. Survey*, Amer. Math. Soc., 1977.

[4] D.H. Fremlin and J. Mendoza, On the integration of vector-valued functions, *Illnois J. Math.* **38** (1994) 127–147.

[5] L. Gordon, Perron integral for derivative in L^r, *Studia. Math.* **28** (1966).

[6] R.A. Gordon, The Denjoy extension of the Bochner, Pettis, and Dunford integrals, *Studia Math.* **92** (1) (1989) 73–91.

[7] R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., 1991.

[8] J.L. Gámez and J. Mendoza, On Denjoy-Dunford and Denjoy-Pettis integrals, *Studia Math.* **130** (2) (1998) 115–133.

[9] T.Y. Lee, Banach valued Henstock-Kurzweil integrable functions are Mcshane integrable on a portion, *Mathematica Bohemica* **4** (2004) 349–354.

[10] G.Q. Liu, On necessary conditions for Henstock integrability, *Real Anal. Exchange* **18** (2) (1992/93) 522–531.

[11] P.M. Musial and Y. Sagher, The L^r Henstock-Kurzweil integral, *Studia Mathematica* **160** (1) (2004) 53–81.

[12] P. Musial, The L_r-variational integral, *Real Analysis Exchange, Summer Symposium* (2010) 96–97.

[13] P. Musial, F. Tulone, Dual of the class of HK_r integrable functions, *Minimax Theory and its Applications* **4** (2019) 151–160.

[14] O. Perron, *Ueber den Integralbegriff*, S.B. Heidelberg, Akad. Wiss, **16**, 1914.

[15] L.D. Piazza, Palermo, V. Marraffa, K. Musiaĺ, Variational Henstock integrability of Banach space valued functions, *Mathematica Bohemica* **141** (2) (2016) 287–296.

[16] De la Vallée Poussin, Intégrales de lebesgue, In: *Functions D'Ensemble, Classes de Baire*, Paris, 1916.

[17] S. Saks, *Theory of Integral*, **2nd** Revised Ed., Hafner, New York, 1937.

[18] S. Schwabik and G.J. Ye, *Topics in Banach Space Integration*, World Scientific, Singapore, 2005.

[19] S.A. Tikare, M.S. Chaudhary, The Henstock-Stieltjes integral in L^r, *Journal of Advanced Research in Pure Mathematics* **1** (2012) 59–80.

[20] G.Y. Ye and T.J. An, On Henstock-Dunford and Henstock-Pettis integrals, *IJMMS* **25** (7) (2001) 467–478.

[21] H. Yoon, J.M. Park, Y.K. Kim, B.M. Kim, The AP-Henstock extension of the Dunford and Pettis integrals, *Journal of the Chungcheong Mathematical Society* **4** (2010) 879–884.