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Group Algebras of Lie Nilpotency Index 15

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Abstract. Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. The classification of group algebras KG with upper Lie nilpotency index $t^L(KG)$ up to 14 has already been done. In this paper, we classify the group algebras KG having upper Lie nilpotency index 15, for $G' = \gamma_2(G)$ as an abelian group.

Keywords: Group algebras; Lie nilpotency index; Lie dimension subgroup.

1. Introduction

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$ and by defining the Lie product $[x, y] = xy - yx$, for all $x, y \in KG$ we have KG considered as a Lie algebra, called the associated Lie algebra of KG and denoted by $L(KG)$. Now inductively, $[x_1, ..., x_n] = [[x_1, ..., x_{n-1}], x_n]$, as $x_1, x_2, ..., x_n \in KG$. Also, the nth lower Lie power $KG^{[n]}$ of KG is an associated ideal generated by all the Lie commutators $[x_1, ..., x_n]$. It is easy to see that $KG^{[1]} = KG$ and nth strong Lie power $KG^{(n)}$ is an associated ideal generated by $[x, y]$, where $x \in KG^{(n-1)}$ and $y \in KG$, $KG^{(1)} = KG$. Also, the group algebra KG is called the Lie nilpotent if $KG^{[n]} = 0$, for some $n \in N$ and called strongly Lie nilpotent if $KG^{(n)} = 0$, for some $n \in N$. The least positive integer n, for which $KG^{[n]} = 0$ and $KG^{(n)} = 0$ is called the Lie nilpotency index (denoted by $t_L(KG)$) and strong Lie nilpotency index (denoted by $t^L(KG)$) of KG respectively.

The notations and basic definitions are same as discussed in [2]. Bhandari and Passi in [1], proved that for a Lie nilpotent group algebra KG, if $p \geq 5$, then $t_L(KG) = t^L(KG)$. But the question whether $t_L(KG) = t^L(KG)$ is still open in general. In [14], it has been proved that if KG is Lie nilpotent, then $t_L(KG) \leq t^L(KG) \leq |G'| + 1$. Thus $|G'| + 1$ is the maximal Lie nilpotency index. Shalev [13], proved that if G is a finite p-group and $CharK = p \geq 5$, then $t_L(KG) = |G'| + 1$ if and only if G' is cyclic. Some other interesting results are given in [3, 4, 5, 7, 8, 9, 10, 11, 12].

Chandra and Sahai [6] classified the strong Lie nilpotent group algebras with $t^L(KG)$ up to 8. Further, strong Lie nilpotency index $t^L(KG)$ up to 13 was characterized by Sharma, Siwach and Sahai in [16, 15]. Recently, Bhatt et al. [2] characterized the group algebras with $t^L(KG)$ equal to 14. In this paper, we have classified the group algebras having upper Lie nilpotency index 15, for G' as an abelian group.

Lemma 1.1. [12] *Let K be a field with $CharK = p > 0$ and G be a nilpotent group such that $|G'| = p^n$ and $exp(G') = p^l$. Then the following statements hold:*

(i) *If $d_{(l+1)} = 0$ for some $l < pm$, then $d_{(pm+1)} \leq d_{(m+1)}$.*
(ii) *If $d_{(m+1)} = 0$, then $d_{(s+1)} = 0$ for all $s \geq m$ with $\vartheta_{p'}(s) \geq \vartheta_{p'}(m)$ where $\vartheta_{p'}(x)$ is the maximal divisor of x which is relatively prime to p.*

2. Main Results

Theorem 2.1. *Let G be a group and K be a field of characteristics $p > 0$ such that KG is Lie nilpotent. Then $t^L(KG) = 15$ if and only if $p = 2$ and one of the following condition satisfied:*

(i) $G' \cong (C_2)^{13}$, $\gamma_3(G) = 1$;
(ii) (a) $G' \cong (C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^6$ *or* $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^6$ *or* $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^7$, $\gamma_3(G) \cong (C_2)^6$, $|G'^2 \cap \gamma_3(G)| = 1$;
(iii) (a) $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^8$, $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$;
(iv) (a) $G' \cong C_8 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ *or* $\gamma_3(G) \cong C_4 \times (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $G' \cong (C_4)^2 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ *or* $\gamma_3(G) \cong C_4 \times (C_2)^2$,

$|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ *or* $\gamma_3(G) \cong C_4 \times (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^6$, $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$;

(v) (a) $G' \cong (C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $G' \cong C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^7$, $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$;

(vi) (a) $G' \cong (C_4)^4 \times C_2$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^4$;
(b) $G' \cong (C_4)^3 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_4)^2 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong C_4 \times (C_2)^7$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $G' \cong (C_2)^9$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(vii) (a) $G' \cong C_8 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $G' \cong C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^7$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(viii) (a) $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(b) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^8$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(viii) (a) $G' \cong (C_4)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;
(b) $G' \cong (C_4)^2 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;
(c) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$
(d) $G' \cong (C_2)^6$, $\gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $G' \cong (C_4)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$;
(f) $G' \cong (C_4)^2 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$;

(g) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$;
(h) $G' \cong (C_2)^6$, $\gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(ix) (a) $G' \cong (C_4)^3 \times (C_2)^4$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$;
(b) $G' \cong (C_4)^2 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ *or* $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong C_4 \times (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^{10}$, $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;

(x) (a) $G' \cong C_8 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(b) $G' \cong (C_4)^2 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(c) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^6$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xi) (a) $G' \cong C_8 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ *or* $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 2$;
(b) $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ *or* $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 2$;
(c) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ *or* $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^8$, $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xii) (a) $G' \cong C_8 \times C_4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 4$;
(b) $G' \cong C_8 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(c) $G' \cong (C_4)^2 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(d) $G' \cong C_4 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $G' \cong (C_2)^5$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xiii) (a) $G' \cong (C_4)^2 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(b) $G' \cong C_4 \times (C_2)^7$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^9$, $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xiv) (a) $G' \cong (C_4)^3 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$,$\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$;
(b) $G' \cong (C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;
(c) $G' \cong C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;

(d) $G' \cong (C_2)^7$, $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $G' \cong (C_4)^3 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong (C_2)^2$;
(f) $(C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$;
(g) $C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$;
(h) $G' \cong (C_2)^7$, $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xv) (a) $G' \cong (C_4)^2 \times (C_2)^7$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$;
(b) $G' \cong C_4 \times (C_2)^9$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$ *or* $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(c) $G' \cong (C_2)^{11}$, $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xvi) (a) $G' \cong C_8 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$; (b) $G' \cong (C_4)^2 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$;
(c) $G' \cong C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$;
(d) $G' \cong (C_2)^7$, $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xvii) (a) $G' \cong C_8 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(b) $G' \cong (C_4)^2 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$;
(c) $G' \cong C_4 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ *or* $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(d) $G' \cong (C_2)^5$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $G' \cong C_8 \times (C_2)^2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$;
(f) $G' \cong (C_4)^2 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$;
(g) $G' \cong C_4 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$;
(h) $G' \cong (C_2)^5$, $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xviii) (a) $G' \cong C_8 \times (C_2)^6$, $\gamma_3(G) \subseteq G'^2 \cong C_4$;
(b) $G' \cong (C_4)^2 \times (C_2)^5$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$;
(c) $G' \cong C_4 \times (C_2)^7$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$;
(d) $G' \cong (C_2)^9$, $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xix) (a) $G' \cong C_4 \times (C_2)^{10}$, $G'^2 = \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$;
(b) $G' \cong (C_2)^{12}$, $G'^2 \cap \gamma_3(G) = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$;

(xx) (a) $G' \cong (C_4)^2 \times (C_2)^6$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$;
(b) $G' \cong C_4 \times (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$;
(c) $G' \cong (C_2)^{10}$, $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xxi) (a) $G' \cong (C_4)^3 \times (C_2)^2$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$, $\gamma_4(G) \cong C_2$;
(b) $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$;
(c) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$,$\gamma_4(G) \cong C_2$;
(d) $G' \cong (C_2)^8$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$;
(e) $(C_4)^3 \times (C_2)^2$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$;
(f) $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$;
(g) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$;

(h) $G' \cong (C_2)^8$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$;

(xxii) (a) $G' \cong (C_4)^3$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$,$\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ *or*$\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$;

(b) $G' \cong (C_4)^2 \times (C_2)^2$, $\gamma_3(G) \subseteq G'^2 \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;

(c) $G' \cong C_4 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ *or* $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$;

(d) $G' \cong (C_2)^6$, $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$.

Proof. Let $t^L(KG) = 15$. Since $p+1 \le t^L(KG)$, $p = 3, 5, 7, 11$ or 13 are not possible. So, we have only $p = 2$.

Let $p = 2$. Since $t^L(KG) = 2 + (p-1)\Sigma m d_{(m+1)}$, so $\Sigma m d_{(m+1)} = 13$ and $d_{(k)} = 0$ for all $k \ge 14$. Let $|G'| = p^n$. If $d_{(3)} = 0$, then $D_{(3),K}(G) = G'^2\gamma_3(G) = 1$. Hence, $d_{(2)} = 13$ and $G' \cong (C_2)^{13}$. Now let $d_{(3)} \ne 0$. Clearly $d_{(2)} \ne 0$ and $d_{(3)} < 7$.

If $d_{(3)} = 6$, then by Lemma 1.1 $d_{(2)} = 1$ and $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$, $|D_{(3),K}(G)| = 2^6$ and $|G'| = 2^7$. In this case, G' is abelian, we have the following possibilities: $G' \cong (C_4)^3 \times C_2$ or $(C_4)^2 \times (C_2)^3$ or $C_4 \times (C_2)^5$ or $(C_2)^7$. Now, if $G' \cong (C_4)^3 \times C_2$, then $D_{(3),K}(G) < 2^6$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^6$ or $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong (C_2)^4$,$|G'^2 \cap \gamma_3(G)| = 1$. If $C_4 \times (C_2)^5$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^6$ or $\gamma_3(G) \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong (C_2)^6$.

Now if $d_{(3)} = 5$, then by Lemma 1.1 $d_{(2)} = 3$ and thus $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$, $|D_{(3),K}(G)| = 2^5$ and $|G'| = 2^8$. In this case G' is abelain, so we have the following possibilities: $G' \cong (C_4)^4$ or $(C_4)^3 \times (C_2)^2$ or $(C_4)^2 \times (C_2)^4$ or $C_4 \times (C_2)^6$ or $(C_2)^8$. Let $G' \cong (C_4)^4$ or $C_4)^3 \times (C_2)^2$. Then $|D_{(3),K}(G) < 2^5$. If $G' \cong (C_4)^2 \times (C_2)^4$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^6$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^8$, then $\gamma_3(G) \cong (C_2)^5$.

Let $d_{(3)} = 4$. Then $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} = 5$. In view of Lemma 1.1, we have following cases: $d_{(5)} = d_{(2)} = 1$ or $d_{(4)} = 1, d_{(2)} = 2$ or $d_{(2)} = 5$.

If $d_{(2)} = d_{(5)} = 1$, $d_{(3)} = 4$, then $|G'| = 2^6$, $|D_{(3),K}(G)| = 2^5$,$D_{(4),K}(G) = D_{(5),K}(G) \cong (C_2)$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is an abelian group, so we have the following possibilities: $G' \cong (C_8)^2$ or $G' \cong C_8 \times C_4 \times C_2$ or $G' \cong C_8 \times (C_2)^3$ or $G' \cong (C_4)^3$ or $G' \cong (C_4)^2 \times (C_2)^2$ or $G' \cong C_4 \times (C_2)^4$ or $G' \cong (C_2)^6$. If $G' \cong (C_8)^2$, then $|G'^2| = 16$ but $|G'^2| \le 4$. Also, if $G' \cong C_8 \times C_4 \times C_2$, then $|G'^2| = 8$ but $|G'^2| \le 4$. If $G' \cong C_8 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ or $\gamma_3(G) \cong C_4 \times (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_4)^3$, then $|G'^2| = 8$ but $|G'^2| \le 4$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ or $\gamma_3(G) \cong C_4 \times (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^3$ or $\gamma_3(G) \cong C_4 \times (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$. If

$G' \cong (C_2)^6$, then $\gamma_3(G) \cong (C_2)^5$.

If $d_{(2)} = 2$, $d_{(4)} = 1$, $d_{(3)} = 4$, then $|G'| = 2^7$, $|D_{(3),K}(G)| = 2^5$, $|D_{(4),K}(G)| = 2$ and $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$. Thus $\gamma_4(G) \cong (C_2)$, $4 \le |\gamma_3(G)| \le 16$. If G' is abelian, So we have the following possibilities: $G' \cong (C_4)^3 \times C_2$ or $(C_4)^2 \times (C_2)^3$ or $C_4 \times (C_2)^5$ or $(C_2)^7$. If $G' \cong (C_4)^3 \times C_2$, here $|G'^2| = 8$ but $|G'^2| \le 4$. Hence this case is not possible. If $G' \cong (C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^5$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong (C_2)^5$.

Now, if $d_{(2)} = 5$, $d_{(3)} = 4$, then $|G'| = 2^9$, $|D_{(3),K}(G)| = 2^4$ and $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$. In this case G' is abelian. So possibilities are: $G' \cong (C_4)^4 \times C_2$ or $(C_4)^3 \times (C_2)^3$ or $(C_4)^2 \times (C_2)^5$ or $C_4 \times (C_2)^7$ or $(C_2)^9$. If $G' \cong (C_4)^4 \times C_2$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^4$. If $G' \cong (C_4)^3 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_4)^2 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^7$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^9$, then $\gamma_3(G) \cong (C_2)^4$.

Let $d_{(3)} = 3$. Then, $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} + 7d_{(8)} = 7$.

Now, let $d_{(8)} = 0$. Then $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} = 7$. Thus we have following cases: $d_{(7)} = d_{(2)} = 1$ or $d_{(6)} = 1$, $d_{(2)} = 2$ or $d_{(5)} = 1$, $d_{(2)} = 3$ or $d_{(5)} = 1$, $d_{(4)} = 1$ or $d_{(4)} = 1$, $d_{(2)} = 4$ or $d_{(4)} = 2$, $d_{(2)} = 1$ or $d_{(2)} = 7$.

Let $d_{(2)} = d_{(7)} = 1$, $d_{(3)} = 3$. Then by Lemma 1.1, $d_{(3+1)} = 0$, $\vartheta_2(6) \ge \vartheta_2(3)$ and so $d_{(7)} = 0$. Thus this case is not possible.

Let $d_{(6)} = 1$, $d_{(2)} = 2$, $d_{(3)} = 3$. Then by Lemma 1.1, $d_{(3+1)} = 0$, $\vartheta_2(5) \ge \vartheta_2(3)$ and so $d_{(6)} = 0$. Thus this case is not possible.

Again by Lemma 1.1, $d_{(5)} = 1$, $d_{(4)} = 1$, $d_{(3)} = 3$ is also not possible.

If $d_{(5)} = 1$, $d_{(2)} = d_{(3)} = 3$, then $|G'| = 2^7$, $|D_{(3),K}(G)| = 2^4$, $D_{(4),K}(G) = D_{(5),K}(G) \cong C_2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. Let G' be abelian. Thus we have the following possibilities: $G' \cong (C_8)^2 \times C_2$ or $G' \cong C_8 \times (C_4)^2$ or $G' \cong C_8 \times C_4 \times (C_2)^2$ or $G' \cong C_8 \times (C_2)^4$ or $G' \cong (C_4)^3 \times C_2$ or $G' \cong C_4 \times (C_2)^5$ or $G' \cong (C_4)^2 \times (C_2)^3$ or $G' \cong (C_2)^7$. If $G' \cong (C_8)^2 \times C_2$, then $|G'^2| = 16$ but $|G'^2| \le 4$. If $G' \cong C_8 \times (C_4)^2$, here $|G'^2| = 16$ but $|G'^2| \le 4$. If $G' \cong C_8 \times C_4 \times (C_2)^2$, here $|G'^2| = 8$ but $|G'^2| \le 4$. If $G' \cong C_8 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_4)^3 \times C_2$, here $|G'^2| = 8$ but $|G'^2| \le 4$. If $G' \cong C_4 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong (C_2)^4$.

Let $d_{(4)} = 1$, $d_{(2)} = 4$, $d_{(3)} = 3$. Then $|G'| = 2^8$, $|D_{(3),K}(G)| = 2^4$, $|D_{(4),K}(G)| = 2$ and $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$. So $\gamma_4(G) \cong C_2$ and $4 \le |\gamma_3(G)| \le 16$. If G' is abelian, then possible choices for G' are $(C_4)^4$ or $(C_4)^3 \times (C_2)^2$ or $(C_4)^2 \times (C_2)^4$ or $C_4 \times (C_2)^6$ or $(C_2)^8$. If $G' \cong (C_4)^4$, then $|G'^2| = 16$ but $|G'^2| \le 4$. If $G' \cong (C_4)^3 \times (C_2)^2$, then $|G'^2| = 8$ but

$|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^6$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^8$, then $\gamma_3(G) \cong (C_2)^4$.

Let $d_{(4)} = 2$, $d_{(2)} = 1$, $d_{(3)} = 3$. Then $|G'| = 2^6$, $|D_{(3),K}(G)| = 2^5$, $|D_{(4),K}(G)| = 2^2$ and $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$, $\gamma_4(G) \cong C_2$ or $(C_2)^2$, $|\gamma_3(G)| = 8$ or 16 or 32. If G' is abelian, then possible choices for G' are: $G' \cong (C_4)^3$ or $(C_4)^2 \times (C_2)^2$ or $C_4 \times (C_2)^4$ or $(C_2)^6$. Let $|\gamma_4(G)| = 2$. Now if $G' \cong (C_4)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_2)^6$, then $\gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong C_2$. Now let $|\gamma_4(G)| = 2^2$. If $G' \cong (C_4)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong (C_2)^6$, then $\gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) \cong (C_2)^2$.

Let $d_{(2)} = 7$, $d_{(3)} = 3$. Then $|G'| = 2^{10}$, $|D_{(3),K}(G)| = 2^3$, $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$. In this case G' is abelian. So we have the following possibilities: $G' \cong (C_4)^5$ or $(C_4)^4 \times (C_2)^2$ or $(C_4)^3 \times (C_2)^4$ or $(C_4)^2 \times (C_2)^6$ or $C_4 \times (C_2)^8$ or $(C_2)^{10}$. Now, if $G' \cong (C_4)^5$, here $|G'^2| = 32$ but $|G'^2| \leq 8$. If $G' \cong (C_4)^4 \times (C_2)^2$, here $|G'^2| = 16$ but $|G'^2| \leq 8$. If $G' \cong (C_4)^3 \times (C_2)^4$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$. If $G' \cong (C_4)^2 \times (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$ or $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong C_4 \times (C_2)^8$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^{10}$, then $\gamma_3(G) \cong (C_2)^3$.

Let $d_{(3)} = 2$. Then, $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} + 7d_{(8)} + 8d_{(9)} + 9d_{(10)} = 9$. If $d_{(10)} = 0$, then $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} + 7d_{(8)} + 8d_{(9)} = 9$. And we have following possibilities for $d_{(8)} \neq 0$: $d_{(9)} = 1$, $d_{(2)} = 1$ and $d_{(8)} = 1$, $d_{(2)} = 2$. But from the Lemma 1.1, these cases are not possible. Now, if $d_{(8)} = 0$, then either $d_{(7)} = 1$, $d_{(4)} = 1$ or $d_{(7)} = 1$,$d_{(2)} = 3$. Again the above cases are discarded by Lemma 1.1.

Now, if $d_{(7)} = 0$, then $d_{(6)} = 1$, $d_{(5)} = 1$ or $d_{(6)} = 1$, $d_{(2)} = 1$, $d_{(4)} = 1$ or $d_{(6)} = 1$, $d_{(2)} = 4$. The cases for $d_{(7)} = 0$ is not possible in view of Lemma 1.1. Now, if $d_{(6)} = 0$, $d_{(3)} = 2$, then we have following possibilities: $d_{(5)} = d_{(4)} = 1$, $d_{(2)} = 2$ or $d_{(5)} = 1$, $d_{(2)} = 5$ or $d_{(5)} = 2$, $d_{(2)} = 1$.

Let $d_{(5)} = d_{(4)} = 1$, $d_{(2)} = d_{(3)} = 2$. Then $|G'| = 2^6$, $|D_{(3),K}(G)| = 2^4$, $|D_{(4),K}(G)| = 2^2$, $|D_{(5),K}(G)| = 2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong (C_8)^2$ or $G' \cong C_8 \times C_4 \times C_2$ or $G' \cong C_8 \times (C_2)^3$ or $G' \cong (C_4)^3$ or $G' \cong (C_4)^2 \times (C_2)^2$ or

$G' \cong C_4 \times (C_2)^4$ or $G' \cong (C_2)^6$. If $G' \cong (C_8)^2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times C_4 \times C_2$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^3$, then, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong (C_4)^3$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^6$, then $\gamma_3(G) \cong (C_2)^4$.

If $d_{(5)} = 1$, $d_{(2)} = 5$, $d_{(3)} = 2$, then $|G'| = 2^8$, $|D_{(3),K}(G)| = 2^3$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = 2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong (C_8)^2 \times C_4$ or $G' \cong (C_8)^2 \times (C_2)^2$ or $G' \cong C_8 \times (C_4)^2 \times C_2$ or $G' \cong C_8 \times C_4 \times (C_2)^3$ or $G' \cong C_8 \times (C_2)^5$ or $G' \cong (C_4)^3 \times (C_2)^2$ or $G' \cong (C_4)^2 \times (C_2)^4$ or $G' \cong C_4 \times (C_2)^6$ or $G' \cong (C_2)^8$. If $G' \cong (C_8)^2 \times C_4$, then $|G'^2| = 32$ but $|G'^2| \leq 4$. If $G' \cong (C_8)^2 \times (C_2)^2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_4)^2 \times C_2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times C_4 \times (C_2)^3$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ or $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong (C_4)^3 \times (C_2)^2$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ or $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong C_4 \times (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$ or $\gamma_3(G) \cong C_4$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^8$, then $\gamma_3(G) \cong (C_2)^3$.

If $d_{(2)} = 1$, $d_{(5)} = d_{(3)} = 2$, then $|G'| = 2^5$, $|D_{(3),K}(G)| = 2^4$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = 4$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong C_8 \times C_4$ or $G' \cong C_8 \times (C_2)^2$ or $G' \cong (C_4)^2 \times C_2$ or $G' \cong C_4 \times (C_2)^3$ or $G' \cong (C_2)^5$. If $G' \cong C_8 \times C_4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 4$. If $G' \cong C_8 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong (C_4)^2 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong C_4 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^5$, then $\gamma_3(G) \cong (C_2)^4$.

Now if $d_{(5)} = 0$, then we have either $d_{(3)} = 2$, $d_{(4)} = 1$, $d_{(2)} = 6$ or $d_{(4)} = 2$, $d_{(2)} = 3$, $d_{(3)} = 2$ or $d_{(2)} = 9$, $d_{(3)} = 2$.

Let $d_{(4)} = 1$, $d_{(2)} = 6$, $d_{(3)} = 2$. Then, $|G'| = 2^9$, $|D_{(3),K}(G)| = 2^3$, $|D_{(4),K}(G)| = 2$ and $G'^4\gamma_3(G)^2\gamma_5(G) = 1$. So $\gamma_4(G) \cong C_2$, $|\gamma_3(G)| = 4$ or 8. If G' is abelian, then possible choices for G' are $(C_4)^4 \times C_2$ or $(C_4)^3 \times (C_2)^3$ or $(C_4)^2 \times (C_2)^5$ or $C_4 \times (C_2)^7$ or $(C_2)^9$. If $G' \cong (C_4)^4 \times C_2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^3 \times (C_2)^3$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^5$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong C_4 \times (C_2)^7$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^9$, then $\gamma_3(G) \cong (C_2)^3$.

Let $d_{(2)} = 3$, $d_{(4)} = d_{(3)} = 2$. Then, $|G'| = 2^7$ and $|D_{(3),K}(G)| = 2^4$, $|D_{(4),K}(G)| = 2^2$ and $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$, $\gamma_4(G) \cong C_2$ or $(C_2)^2$, $|\gamma_3(G)| = 4$ or 8 or 16. If G' is abelian, then possible choices for G' are: $G' \cong (C_4)^3 \times C_2$ or $(C_4)^2 \times (C_2)^3$ or $C_4 \times (C_2)^5$ or $(C_2)^7$. First, let $|\gamma_4(G)| = 2$. This implies $|\gamma_3(G)| = 2^3$ or 2^3 or 2^4. Now, if $G' \cong (C_4)^3 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$,$\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong C_4 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$.

Now, let $|\gamma_4(G)| = 2^2$. This implies $|\gamma_3(G)| = 2^3$ or 2^4. If $G' \cong (C_4)^3 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong (C_2)^2$. If $(C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong (C_2)^2$. If $C_4 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$.

Let $d_{(2)} = 9$, $d_{(3)} = 2$. Then $|G'| = 2^{11}$, $|D_{(3),K}(G)| = 2^2$, $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$. In this case G' is abelian. So we have the following possibilities: $G' \cong (C_4)^5 \times C_2$ or $(C_4)^4 \times (C_2)^3$ or $(C_4)^3 \times (C_2)^5$ or $(C_4)^2 \times (C_2)^7$ or $C_4 \times (C_2)^9$ or $(C_2)^{11}$. Now, if $G' \cong (C_4)^5 \times C_2$, then $|G'^2| = 32$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^4 \times (C_2)^3$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^3 \times (C_2)^5$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^7$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^9$, then either $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$ or $\gamma_3(G) \cong C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^{11}$, then $\gamma_3(G) \cong (C_2)^2$.

Let $d_{(3)} = 1$. Then, $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} + 7d_{(8)} + 8d_{(9)} + 9d_{(10)} + 10d_{(11)} + 11d_{(12)} = 11$.

Let $d_{(12)} = 0$. Then, $d_{(2)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} + 7d_{(8)} + 8d_{(9)} + 9d_{(10)} + 10d_{(11)} = 11$. Then, we have following possibilities: $d_{(11)} = 1$, $d_{(2)} = 1$. Now, if $d_{(11)} = 0$, then $d_{(10)} = 1$, $d_{(2)} = 2$. Thus all these cases are discarded by Lemma 1.1. Let $d_{(10)} = 0$. Then we have either $d_{(9)} = 1$, $d_{(4)} = 1$ or $d_{(9)} = 1$, $d_{(2)} = 3$. Then all the above cases are discarded by Lemma 1.1. Now if $d_{(9)} = 0$, then we have $d_{(8)} = 1$, $d_{(5)} = 1$ or $d_{(8)} = 1$, $d_{(2)} = d_{(4)} = 1$ or $d_{(8)} = 1$, $d_{(2)} = 4$. Then all the above cases are discarded by Lemma 1.1. Now, if $d_{(8)} = 0$, then we have $d_{(7)} = 1$, $d_{(6)} = 1$ or $d_{(7)} = 1$, $d_{(2)} = d_{(5)} = 1$ or $d_{(7)} = 1$, $d_{(4)} = 1$, $d_{(2)} = 2$. Then all the above cases are discarded by Lemma 1.1. Now, if $d_{(7)} = 0$, then we have $d_{(6)} = 1$, $d_{(5)} = 1$, $d_{(2)} = 2$ or $d_{(6)} = d_{(4)} = 1$, $d_{(2)} = 3$ or $d_{(6)} = 1$, $d_{(4)} = 2$ or $d_{(6)} = 1$, $d_{(2)} = 6$ or $d_{(6)} = 2$, $d_{(2)} = 1$. Then, all the above cases are discarded by Lemma 1.1. Now, if $d_{(6)} = 0$, then we have $d_{(5)} = d_{(4)} = 1$, $d_{(2)} = 4$ or $d_{(5)} = d_{(2)} = 1$, $d_{(4)} = 2$, or $d_{(5)} = 1$, $d_{(2)} = 7$ or $d_{(5)} = 2$, $d_{(2)} = 3$ or $d_{(5)} = 2$, $d_{(4)} = 1$.

Let $d_{(2)} = 4$, $d_{(5)} = d_{(4)} = d_{(3)} = 1$. Then $|G'| = 2^7$, $|D_{(3),K}(G)| = 2^3$, $|D_{(4),K}(G)| = 2^2$, $|D_{(5),K}(G)| = 2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong (C_8)^2 \times C_2$ or $G' \cong C_8 \times (C_4)^2$ or $G' \cong C_8 \times C_4 \times (C_2)^2$ or $G' \cong C_8 \times (C_2)^4$ or $G' \cong (C_4)^3 \times C_2$ or $G' \cong C_4 \times (C_2)^5$ or $G' \cong (C_4)^2 \times (C_2)^3$ or $G' \cong (C_2)^7$. If $G' \cong (C_8)^2 \times C_2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_4)^2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times C_4 \times (C_2)^2$, here $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If $G' \cong (C_4)^3 \times C_2$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If $G' \cong C_4 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$. If $G' \cong (C_2)^7$, then $\gamma_3(G) \cong C_4 \times C_2$.

Let $d_{(4)} = 2$, $d_{(2)} = d_{(5)} = d_{(3)} = 1$. Then $|G'| = 2^5$, $|D_{(3),K}(G)| = 2^4$,

$|D_{(4),K}(G)| = 2^3$, $|D_{(5),K}(G)| = 2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong C_8 \times C_4$ or $G' \cong C_8 \times (C_2)^2$ or $G' \cong (C_4)^2 \times C_2$ or $G' \cong C_4 \times (C_2)^3$ or $G' \cong (C_2)^5$. First, let $|\gamma_4(G)| = 2^2$. This implies $|\gamma_3(G)| = 2^3$ or 2^4. Now, if $G' \cong C_8 \times C_4$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong (C_4)^2 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$. If $G' \cong C_4 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$ or $\gamma_3(G) \cong C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$. If $G' \cong (C_2)^5$, then $\gamma_3(G) \cong (C_2)^4$. Now, let $|\gamma_4(G)| = 2^3$. This implies $|\gamma_3(G)| = 2^4$. If $G' \cong C_8 \times C_4$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$. If $G' \cong (C_4)^2 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$. If $G' \cong C_4 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times (C_2)^2$. If $G' \cong (C_2)^5$, then $\gamma_3(G) \cong (C_2)^4$.

Let $d_{(2)} = 7$, $d_{(5)} = d_{(3)} = 1$. Then $|G'| = 2^9$, $|D_{(3),K}(G)| = 2^2$, $|D_{(4),K}(G)| = |D_{(5),K}(G)| = 2$ and $D_{(6),K}(G) = G'^8\gamma_3(G)^4\gamma_4(G)^2\gamma_6(G) = 1$. If G' is abelian, then we have the following possibilities: $G' \cong (C_8)^3$ or $G' \cong (C_8)^2 \times C_4 \times C_2$ or $G' \cong C_8 \times (C_4)^3$ or $G' \cong C_8 \times (C_4)^2 \times (C_2)^2$ or $G' \cong C_8 \times (C_2)^6$ or $G' \cong (C_4)^4 \times C_2$ or $G' \cong (C_4)^3 \times (C_2)^3$ or $G' \cong (C_4)^2 \times (C_2)^5$ or $G' \cong C_4 \times (C_2)^7$ or $G' \cong (C_2)^9$. If $G' \cong (C_8)^3$, then $|G'^2| = 64$ but $|G'^2| \leq 4$. If $G' \cong (C_8)^2 \times C_4 \times C_2$, then $|G'^2| = 32$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_4)^3$, then $|G'^2| = 32$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_4)^2 \times (C_2)^2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong C_8 \times (C_2)^6$, then $\gamma_3(G) \subseteq G'^2 \cong C_4$. If $G' \cong (C_4)^4 \times C_2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^3 \times (C_2)^3$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^5$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^7$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$. If $G' \cong (C_2)^9$, then $\gamma_3(G) \cong (C_2)^2$.

Let $d_{(2)} = 3$, $d_{(5)} = 2$, $d_{(3)} = 1$. Then, by Lemma 1.1 as $d_{(3+1)} = 0$, $d_{(4+1)} \leq d_{(2+1)}$ implies $2 \leq 1$, which is not possible. Let $d_{(5)} = 2$, $d_{(4)} = d_{(3)} = 1$. By Lemma 1.1, this case is also not possible. Let $d_{(5)} = 0$. Then we have $d_{(4)} = 1$, $d_{(2)} = 8$ or $d_{(4)} = 2$, $d_{(2)} = 5$ or $d_{(4)} = 3$, $d_{(2)} = 2$ or $d_{(2)} = 11$.

Let $d_{(2)} = 11$, $d_{(3)} = 1$. Then, $|G'| = 2^{12}$, $|D_{(3),K}(G)| = 2$, $D_{(4),K}(G) = G'^4\gamma_3(G)^2\gamma_4(G) = 1$ and exponent of G' is at most 4 and $|G'^2| \leq 2$. Now G' is abelian in this case. Thus possible G' are: $C_4 \times (C_2)^{10}$ or $(C_2)^{12}$. If $G' \cong C_4 \times (C_2)^{10}$, then $G'^2 = \gamma_3(G) \cong C_2$ and $\gamma_4(G) = 1$. If $G' \cong (C_2)^{12}$, then $G'^2 \cap \gamma_3(G) = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$.

Let $d_{(2)} = 8$, $d_{(4)} = d_{(3)} = 1$. Then, $|G'| = 2^{10}$, $|D_{(3),K}(G)| = 2^2$, $|D_{(4),K}(G)| = 2$ and $D_{(5),K}(G) = G'^4\gamma_3(G)^2\gamma_5(G) = 1$. So $\gamma_4(G) \cong C_2$, $|\gamma_3(G)| = 4$. If G' is abelian, then possible choices for G' are $(C_4)^5$ or $(C_4)^4 \times (C_2)^2$ or $(C_4)^3 \times (C_2)^4$ or $(C_4)^2 \times (C_2)^6$ or $C_4 \times (C_2)^8$ or $(C_2)^{10}$. If $G' \cong (C_4)^5$, then $|G'^2| = 32$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^4 \times (C_2)^2$, then $|G'^2| = 16$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^3 \times (C_2)^4$, then $|G'^2| = 8$ but $|G'^2| \leq 4$. If $G' \cong (C_4)^2 \times (C_2)^6$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^8$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$. If $G' \cong (C_2)^{10}$, then $\gamma_3(G) \cong (C_2)^2$.

Let $d_{(2)} = 5$, $d_{(4)} = 2$, $d_{(3)} = 1$. Then $|G'| = 2^8$, $|D_{(3),K}(G)| = 2^3$, $|D_{(4),K}(G)| = 2^2$ and $G'^4\gamma_3(G)^2\gamma_5(G) = 1$, $\gamma_4(G) \cong C_2$ or $(C_2)^2$, $|\gamma_3(G)| = 4$ or 8. If G' is abelian, then possible choices for G' are: $G' \cong (C_4)^4$ or $(C_4)^3 \times (C_2)^2$

or $(C_4)^2 \times (C_2)^4$ or $C_4 \times (C_2)^6$ or $(C_2)^8$. First, let $|\gamma_4(G)| = 2$. This implies $|\gamma_3(G)| = 2^2$ or 2^3. Now, if $G' \cong (C_4)^4$, then this case is not possible. If $G' \cong (C_4)^3 \times (C_2)^2$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$. If $G' \cong C_4 \times (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$,$\gamma_4(G) \cong C_2$. If $G' \cong (C_2)^8$, then $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$.

Now, $|\gamma_4(G)| = 2^2$ implies $|\gamma_3(G)| = 2^3$. If $G' \cong (C_4)^4$, then this case is not possible. If $(C_4)^3 \times (C_2)^2$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong C_4 \times (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$. If $G' \cong (C_2)^8$, then $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$.

Let $d_{(2)} = 2$, $d_{(4)} = 3$, $d_{(3)} = 1$. Then $|G'| = 2^6$, $|D_{(3),K}(G)| = 2^4$, $|D_{(4),K}(G)| = 2^3$ and $G'^4\gamma_3(G)^2\gamma_5(G) = 1$. $\gamma_4(G) \cong C_2$ or $(C_2)^2$ or $(C_2)^3$, $|\gamma_3(G)| = 4$ or 8 or 16. If G' is abelian, then possible choices for G' are: $G' \cong (C_4)^3$ or $(C_4)^2 \times (C_2)^2$ or $C_4 \times (C_2)^4$ or $(C_2)^6$. If $G' \cong (C_4)^3$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$,$\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_4(G) \cong C_2$ or$\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_4)^2 \times (C_2)^2$, then $\gamma_3(G) \subseteq G'^2 \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong C_4 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ or $\gamma_3(G) \cong (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$. If $G' \cong (C_2)^6$, then $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ and $|G'^2 \cap \gamma_3(G)| = 1$.

Converse can be easily done by computing $d_{(m)}$'s in each case. ■

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