Southeast Asian Bulletin of Mathematics (2024) 48: 273–284

Southeast Asian
Bulletin of
Mathematics

Category L-Slice and its Properties

K.S. Sabna
Department of Mathematics, K.K.T.M Government College, Pullut, Kerala, India
Email: sabnaks7@gmail.com

N.R. Mangalambal
Department of Mathematics, St. Joseph's College, Irinjalakuda-680121, India
Email: thottuvai@gmail.com

Received 15 August 2021
Accepted 7 May 2022

Communicated by Y.Q. Chen

AMS Mathematics Subject Classification(2020): 06D22, 54A05, 06A12, 03G10

Abstract. The notion of an action σ of a locale L on a join semilattice J with bottom element 0_J is developed and utilized to form the entity (σ, J), which we call L-slice, has properties that could be studied algebraically as well as topologically. In this study, the existence of a contravariant functor from a subcategory of the category L-slice, of L-slices and L-slice homomorphisms, into the category TopWMod, of topological weak modules and continuous weak module homomorphisms, has been established.

Keywords: Weak L-module; Weak L-module homomorphism; Weak L-submodule and topological weak modules.

1. Introduction

From 1914 onwards, topology on a topological space was known as lattice of open subsets. The interrelation between topology and lattice theory was first studied by M.H. Stone in [12, 13, 14]. After his work on topological representation of Boolean algebras and distributive lattices , the relation between topology and lattice theory began to be explored. In [4], complete lattice satisfying infinite distributive law is expressed as pointless topology. Afterwards most of the topological ideas were studied in the localic background. Dual to the notion of theory of locales, we have theory of frames. The study using frame theory is more algebraic and those in localic background are topological.

The action of residuated structure on posets has been studied by many authors, especially in connection with mathematical logic. Such structures are involved in the development of several interesting theories including image processing [9]. In this paper, we have introduced the concept of action of a locale L on a join semilattice J, so that J will inherit some topological properties from the locale L.

Given a complete semiring $(L, +, ., 0_L, 1_L)$, where finite product $\cdot$ distributes over infinite sum $+$ and a monoid $(M, *, 0_M)$, a weak L-module is introduced to be an action of L on $(M, *, 0_M)$. We have defined weak L-module homomorphism between two weak L-modules (δ, M) and (γ, N). It is proved that if $(N, *')$ is commutative, then the collection of all weak L-module homomorphisms from (δ, M) to (γ, N) is a weak L-module.

It is known that there is a contravariant functor from the category of join semilattices with 0 and semilattice homomorphisms, to the category of idempotent topological monoids and continuous monoid homomorphisms. This is a matter of interest to investigate that to which algebraic structure the category of L-slices and L-slice homomorphisms can be associated. In this paper, we have proved that there is a contravarient functor from a subcategory of the category L-slice of L-slices to the category TopWMod of weak topological L-modules.

2. Preliminaries

In this section, we mainly introduce some notions, notations and basic properties which will be used in the rest of the paper.

Definition 2.1. [7] *A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup\{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.*

Example 2.2. [7]

(i) The lattice of open subsets of topological space.
(ii) The Boolean algebra B of all open subsets U of real line R such that U=int(cl(U)).

Definition 2.3. [7] *A map $f : L \to M$ between frames L, M preserving all finite meets (including the top 1) and all joins (including the bottom 0) is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.*

Remark 2.4. The category of frames and frame homomorphisms is denoted by Frm. The opposite of category Frm is the category Loc of locales. We can represent the morphism in Loc as the infima -preserving $f : L \to M$ such that the corresponding left adjoint $f^* : M \to L$ preserves finite meet.

Definition 2.5. [5] *A subset I of a locale L is said to be an ideal if*

(i) *I is a sub-join-semilattice of L; that is $0_L \in I$ and $a \in I, b \in I$ implies $a \sqcup b \in I$; and*

(ii) *I is a lower set; that is $a \in I$ and $b \sqsubseteq a$ imply $b \in I$.*

Definition 2.6. *A semiring is a triple $(S, +, \cdot)$, where S is a set and $+$ and $\cdot$ are binary operations, such that $+$ is commutative, both $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws hold for all $x, y, z \in S$.*

(i) $x.(y + z) = x.y + x.z$

(ii) $(x + y).z = (x.z) + (y.z)$

If $(S, \cdot)$ is a monoid, then $(S, +, \cdot)$ is a semiring with 1.

Definition 2.7. *A complete semiring is a semiring for which the addition monoid is a complete monoid and the following infinitary distributive laws hold $\Sigma(a.a_i) = a.\Sigma a_i$ and $\Sigma(a_i.a) = (\Sigma a_i).a$.*

Definition 2.8. *A topological semiring is a semiring S togather with a topology under which the semiring operations are continuous.*

Definition 2.9. [3] *A category C consist of:*

(i) *A class $Ob\ C$ of objects (notation: $A, B, C, ...$).*

(ii) *A class $Mor\ C$ of morphisms (notation: $f, g, h...$). Each morphism f has a domain or source A (notation: $dom(f)$) and a codomain or target B (notation: $codom(f)$) which are objects of C; this is indicated by writing $f : A \to B$.*

(iii) *A composition law that assign to each pair (f, g) of morphisms satisfying $dom(g) = codom(f)$, a morphism $g \circ f : dom(f) \to codom(g)$ satisfying*

(a) *$h \circ (g \circ f) = (h \circ g) \circ f$ whenever the compositions are defined.*

(b) *For each object A of C there is an identiy $id_A : A \to A$ such that $f \circ id_A = f$ and $id_A \circ g = g$ whenever the composition is defined.*

Definition 2.10. [3] *A category B is said to be a subcategory of the category C provided that the following conditions are satisfied:*

(i) *$Ob(B) \subseteq Ob(C)$.*

(ii) *$Mor(B) \subseteq Mor(C)$.*

(iii) *The domain, codomain and composition functions of B are restriction of the corresponding functions of C.*

(iv) *Every B-identity is a C-identity.*

Definition 2.11. [3] *A morphism $f : A \to B$ in a category C is said to be section in C provided that there exists some C-morphism $g : B \to A$ such that $g \circ f = id_A$.*

Definition 2.12. [3] *A morphism $f : A \to B$ in a category C is said to be a retraction in C provided that there exists some C -morphism $g : B \to A$ such that $f \circ g = id_B$.*

Definition 2.13. [3] *A C-morphism is said to be an isomorphism in C provided that it is both C-section and C-retraction.*

Definition 2.14. [3] *Let C be a category and $A, B \in Obj(C)$. A morphism $f : A \to B$ is epimorphism if $f \circ g = f \circ h$ implies $g = h$ for all morphisms $g, h : B \to C$.*

Definition 2.15. [3] *A C -morphism $f : A \to B$ is said to be a monomorphism in C provided that for all C-morphisms h and k such that $f \circ h = f \circ k$, it follows that $h = k$.*

3. L-Slices

This section discusses the concept of L-slice and some of its properties.

Definition 3.1. [10] *Let L be a locale and $(J, \vee)$ be join semilattice with bottom element 0_J. By the "action of L on J" we mean a function $\sigma : L \times J \to J$ such that the following conditions are satisfied:*

(i) $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ *for all* $a \in L, x_1, x_2 \in J$.
(ii) $\sigma(a, 0_J) = 0_J$ *for all* $a \in L$.
(iii) $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ *for all* $a, b \in L, x \in J$.
(iv) $\sigma(1_L, x) = x$ *and* $\sigma(0_L, x) = 0_J$ *for all* $x \in J$.
(v) $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ *for* $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice.

Proposition 3.2. [10] *Let L be a locale and S a set of order preserving maps $L \to L$ such that:*

(i) *The constant map $0 \in S$ (0 takes everything to 0).*
(ii) *If $f, g \in S$, then $f \vee g \in S$.*
(iii) *For all $a \in L$ and for all $f \in S$, the meet of the constant map a and f is in S (i.e. $f \sqcap a \in S$).*

Then the map $\sigma : L \times S \to S$ defined by $\sigma(a, f)(x) = f(x) \sqcap a$ is an action of L on S.

Example 3.3. [10]

(i) Let L be a locale and I any ideal of L. Consider each $x \in I$ as constant map $x : L \to L$. Then by Proposition 3.2, (σ, I) is an L-slice. In particular (σ, L) is an L-slice.

(ii) Let the locale L be a chain with Top and Bottom elements and J be any join semilattice with bottom element. Define $\sigma : L \times J \to J$ by $\sigma(a, j) = j$ for all $a \neq 0$ and $\sigma(0_L, j) = 0_J$. Then σ is an action of L on J and (σ, J) is an L-slice.

Definition 3.4. [10] *Let (σ, J) be an L-slice of a locale L. A subjoin semilattice J' of J is said to be L-subslice of (σ, J) if J' is closed under action by elements of L.*

Example 3.5.

(i) Let L be a locale and $O(L)$ denote the collection of all order preserving maps on L. Then $(\sigma, O(L))$ is an L-slice, where $\sigma : L \times O(L) \to O(L)$ is defined by $\sigma(a, f) = f_a$, and $f_a : L \to L$ is defined by $f_a(x) = f(x) \sqcap a$. Let $K = \{f \in O(L) : f(x) \sqsubseteq x, \forall x \in L\}$. Then (σ, K) is an L-subslice of the L-slice $(\sigma, O(L))$.

(ii) Let (σ, J) be an L-slice and let $x \in (\sigma, J)$. Define $\langle x \rangle = \{\sigma(a, x); a \in L\}$. Then $(\sigma, \langle x \rangle)$ is an L-subslice of (σ, J) and it is the smallest L-subslice of (σ, J) containing x.

Definition 3.6. [10] *A subslice (σ, I) of an L-slice (σ, J) is said to be ideal of (σ, J) if $x \in (\sigma, I)$ and $y \in (\sigma, J)$ are such that $y \leq x$, then $y \in (\sigma, I)$.*

Definition 3.7. [10] *An ideal (σ, I) of an L-slice(σ, J) is a prime ideal if it has the following properties:*

(i) *If a and b are any two elements of L such that $\sigma(a \sqcap b, x) \in I$, then either $\sigma(a, x) \in I$ or $\sigma(b, x) \in I$.*

(ii) *(σ, I) is not equal to the whole slice (σ, J).*

Definition 3.8. [10] *Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if*

(i) *$f(x_1 \vee x_2) = f(x_1) \vee' f(x_2)$ for all $x_1, x_2 \in (\sigma, J)$.*

(ii) *$f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.*

Example 3.9. [10]

(i) Let (σ, J) be an L-slice and (σ, J') an L-subslice of (σ, J). Then the inclusion map $i : (\sigma, J') \to (\sigma, J)$ is an L-slice homomorphism.

(ii) Let $I = \downarrow(a), J = \downarrow(b)$ be principal ideals of the locale L. Then (σ, I), (σ, J) are L-slices. Then the map $f : (\sigma, I) \to (\sigma, J)$ defined by $f(x) = x \sqcap b$ is an L-slice homomorphism.

Some simple properties of L-slice homomorphism are discussed in the following propositions.

Proposition 3.10. *If $f : (\sigma, J) \to (\mu, K)$ is an L-slice homomorphism, then $f(0_J) = 0_K$.*

Proposition 3.11. *The composition of two L-slice homomorphisms of a locale L is an L-slice homomorphism.*

Proposition 3.12. *Let $(\sigma, J), (\mu, K)$ be L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. If (μ, I) is an ideal of (μ, K), then $(\sigma, f^{-1}(I))$ is an ideal of (σ, J). In particular if (μ, I) is prime, then $(\sigma, f^{-1}(I))$ is also prime.*

Proposition 3.13. *Let (σ, I), (μ, K) be L-slices and let $f : (\sigma, J) \to (\mu, K)$ be a bijective L-slice homomorphism. Then the map $f^{-1} : (\mu, K) \to (\sigma, J)$ is an L-slice homomorphism.*

Proposition 3.14. *Let $(\sigma, J), (\mu, K)$ be L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an injective L-slice homomorphism. Then for any $z \in (\mu, K), f^{-1}(\downarrow z) = \downarrow f^{-1}(z)$.*

Proof. Let $y \in f^{-1}(\downarrow z)$. Then $f(y) \in \downarrow z$. So $f(y) \leq z$ or $y \leq f^{-1}(z)$. Thus $y \in \downarrow f^{-1}(z)$. Hence $f^{-1}(\downarrow z) \subseteq \downarrow f^{-1}(z)$. Now let $y \in \downarrow f^{-1}(z)$. Then $y \leq f^{-1}(z)$ or $f(y) \leq z$. Hence $f(y) \in \downarrow z$ and so $y \in f^{-1}(\downarrow z)$. Hence $f^{-1}(\downarrow z) \subseteq f^{-1}(\downarrow z)$. ■

Definition 3.15. [10] *Let $(\sigma, J), (\mu, K)$ be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be isomorphism if*
(i) *f is one-one,*
(ii) *f is onto,*
(iii) *f is an L-slice homomorphism.*

Proposition 3.16. [10] *Let $(\sigma, J), (\mu, K)$ be L-slices of a locale L. Then the collection L- Hom(J,K) of all L-slice homomorphisms from (σ, J) to (μ, K) is an Lslice.*

Let L-slice denote the category of L-slices and L-slice homomorphisms.

Proposition 3.17. *Let $f : (\sigma, J) \to (\mu, K)$ be an injective L-slice homomorphism. If image $imf = \downarrow z$, where $z \in (\mu, K)$ is a maximal element of (μ, K), then f is a section in the category L-slice.*

Proof. Define $g : (\mu, K) \to (\sigma, J)$ as follows. Let $y \in (\mu, K)$. If $y \in imf$, then $y = f(x)$ for a unique $x \in (\sigma, J)$. Then define $g(y) = x$. If $y \notin imf$, define $g(y) = 0_J$. Then $g : (\mu, K) \to (\sigma, J)$ is an L-slice homomorphism and $(g \circ f)(x) = x$, for all $x \in (\sigma, J)$. Hence f is a section in the category L-slice. ■

Proposition 3.18. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is a retraction in the category L-slice if and only if f is onto.*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be a retraction in the category L-slice. Let $y \in (\mu, K)$. Since $f : (\sigma, J) \to (\mu, K)$ is a retraction, there is an L-slice homomorphism $g : (\mu, K) \to (\sigma, J)$ such that $(f \circ g) = I$. Hence $f(g(y)) = y$ and so f is onto.

Conversely let $f : (\sigma, J) \to (\mu, K)$ be an onto L-slice homomorphism. For each $y \in (\mu, K)$, there is some $x \in (\sigma, J)$ such that $y = f(x)$. Define $g : (\mu, K) \to (\sigma, J)$ by $g(f(x)) = x$. Then we have g is an L-slice homomorphism and $(f \circ g)(y) = y$, for all $y \in (\mu, K)$. Hence f is a retraction in the category L-slice. ■

In a similar manner we can show the following propositions.

Proposition 3.19. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is a monomorphism in the category L-slice if and only if f is injective.*

Proposition 3.20. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is an epimorphism in the category L-slice if and only if f is surjective.*

Proposition 3.21. [10] *Let (σ, J) be an L-slice of a locale L. For each $a \in L$, let $\sigma_a : (\sigma, J) \to (\sigma, J)$ be defined by $\sigma_a(x) = \sigma(a, x)$. Then σ_a is an L-slice homomorphism.*

Proposition 3.22. [10]

(i) *$\sigma_a(x) \leq x$ for all $x \in J$.*

(ii) *If I is an ideal in (σ, J), then $\sigma_a(I)$ is a subslice of (σ, J) and $\sigma_a(I) \subseteq I$.*

4. Weak L-module

Definition 4.1. *Let $(L, +, ., 0_L, 1_L)$ be a complete semiring where finite . distribute over infinite + and let $(M, *, 0_M)$ be a monoid. By an action of L on M, we mean a function $\delta : L \times M \to M$ such that the following conditions are satisfied:*

(i) *$\delta(r + s, x) = \delta(r, x) * \delta(s, x)$ for all $r, s \in L, x \in M$.*

(ii) *$\delta(r, x * y) = \delta(r, x) * \delta(r, y)$.*

(iii) *$\delta(r, 0_M) = 0_M$.*

(iv) *$\delta(r.s, x) = \delta(r, \delta(s, x))$.*

(v) *$\delta(0_L, x) = 0_M$ and $\delta(1_L, x) = x$.*

If δ is an action of L on M, we call (δ, M) as a weak L-module.

Note 4.2. If $(L, \cdot)$ is commutative, then $\delta(r.s, x) = \delta(r, \delta(s, x)) = \delta(s, \delta(r, x))$.

Every L-slice is an example for weak L-module.

Definition 4.3. *Let (δ, M) be a weak L-module. A submonoid M' of M is said to be a weak L-submodule of (δ, M) if M' is closed under action by elements of L.*

Definition 4.4. *A weak L-module homomorphism between weak L-modules $(\delta, M), (\gamma, N)$ is a map $g : (\delta, M) \to (\gamma, N)$ such that*

(i) *$g(x * y) = g(x) *' g(y)$.*

(ii) *$g(\delta(r, x)) = \gamma(r, g(x))$ for all $x, y \in M, r \in L$.*

Proposition 4.5. *Composition of two weak L-module homomorphisms is a weak L-module homomorphism.*

Proposition 4.6. *Let $(\delta, M), (\gamma, N)$ be two weak L-modules. Then the following statements hold:*

(i) *The map $0 : (\delta, M) \to (\gamma, N)$ defined by $0(x) = 0_N$ for all $x \in (\delta, M)$ is a weak L-module homomorphism.*

(ii) *If $f, g : (\delta, M) \to (\gamma, N)$ are two weak L-module homomorphisms and $(N, *')$ is commutative, then $f * g : (\delta, M) \to (\gamma, N)$ defined by $(f * g)(x) = f(x) *' g(x)$ is a weak L-module homomorphism.*

(iii) *If $f : (\delta, M) \to (\gamma, N)$ is a weak L-module homomorphism, then for any $r \in S$, the map $\eta(r, f) : (\delta, M) \to (\gamma, N)$ defined by $\eta(r, f)(x) = \gamma(r, f(x))$ is a weak L-module homomorphism.*

Proposition 4.7. *Let $(\delta, M), (\gamma, N)$ be two weak L-modules, where $(N, *')$ is commutative. Then the collection Δ of all weak L-module homomorphisms from (δ, M) to (γ, N) is weak L-module.*

Proof. For any $f, g \in \Delta$, define $f*g : (\delta, M) \to (\gamma, N)$ by $(f*g)(x) = f(x)*'g(x)$. Then $(\Delta, *)$ is a monoid. Define $\eta : L \times \Delta \to \Delta$ as follows. For each $r \in L, f \in \Delta$, $\eta(r, f) : (\delta, M) \to (\gamma, N)$ is a map defined by $\eta(r, f)(x) = \gamma(r, f(x))$. Then η is an action of L on Δ. Let $r, s \in L, x \in (\delta, M)$. Then we have

(i) $$\begin{aligned}\eta(r + s, f)(x) &= \gamma(r + s, f(x)) = \gamma(r, f(x)) *' \gamma(s, f(x)) \\ &= \eta(r, f)(x) *' \eta(s, f)(x) = (\eta(r, f) * \eta(s, f))(x).\end{aligned}$$

(ii) $$\begin{aligned}\eta(r, f * g)(x) &= \gamma(r, (f * g)(x)) = \gamma(r, f(x) *' g(x)) \\ &= \gamma(r, f(x)) *' \gamma(r, g(x)) = (\eta(r, f) * \eta(r, g))(x).\end{aligned}$$

(iii) $\eta(r,0)(x) = \gamma(r,0(x)) = \gamma(r, 0_N) = 0_N = 0(x)$.

(iv) $\eta(r.s, f)(x) = \gamma(r.s, f(x)) = \gamma(r, \gamma(s, f(x)))$
$= \gamma(r, \eta(s, f)(x)) = \eta(r, \eta(s, f))(x)$.

(v) $\eta(0_L, f)(x) = \gamma(0_L, f(x)) = 0_N = 0(x)$.

Hence (η, Δ) is a weak L-module homomorphism. ■

Proposition 4.8. *Let $f : (\delta, M) \to (\gamma, N)$ be a weak L-module homomorphism. Then the following statements hold:*

(i) *$kerf = \{x \in (\delta, M) : f(x) = 0_N\}$ is a weak L-submodule of (δ, M).*

(ii) *$imf = \{y \in (\gamma, N) : y = f(x) for \ some \ x \in (\delta, M)\}$ is a weak L-submodule of (γ, N).*

Proposition 4.9. *Let $f : (\delta, M) \to (\delta, M)$ be a weak L-module homomorphism. Then $F = \{x \in (\delta, M) : f(x) = x\}$ is a weak L-submodule.*

Proof. Let $f : (\delta, M) \to (\delta, M)$ be a weak L-module homomorphism. Since $f(0_M) = 0_M$, $0_M \in F$. Let $x, y \in F$. Then $f(x * y) = f(x) * f(y) = x * y$. Hence $x * y \in F$. Thus $(F, *)$ is a submonoid of $(M, *)$. Let $r \in L, x \in F$. Then $f(\delta(r, x)) = \delta(r, f(x)) = \delta(r, x)$. Thus $\delta(r, x) \in F$. Hence (δ, F) is a weak L-submodule of (δ, M). ■

Definition 4.10. *A topological weak L-module is a triplet (δ, M, τ), where τ is a topology on (δ, M) such that*

(i) *$* : M \times M \to M$ is continuous.*

(ii) *$\delta_a : M \to M$ defined by $\delta_a(x) = \delta(a, x)$ is continuous for every $a \in L$.*

Definition 4.11. *A morphism between topological weak L-module (δ, M, τ_1), (γ, N, τ_2) is a map $h : (\delta, M, \tau_1) \to (\gamma, N, \tau_2)$ such that*

(i) *$h(x * y) = h(x) *' h(y)$ for all $x, y \in M$.*

(ii) *$h(\delta(a, x)) = \gamma(a, h(x))$ for all $x \in M, a \in L$.*

(iii) *h is continuous.*

5. Topological Weak L-module Associated with L-slice

Let (σ, J) be an L-slice with bottom element 0. Let $Pt(J) = \{\downarrow x : x \in (\sigma, J)\}$. Define a binary operation $*$ on $Pt(J)$ by $\downarrow x * \downarrow y = \downarrow x \vee y$. Then $(Pt(J), *, 0)$ is a commutative monoid. Define $\delta : L \times Pt(J) \to Pt(J)$ by $\delta(a, \downarrow x) = \downarrow \sigma(a, x)$. In the next proposition, we will show that δ is an action of the semiring L on the monoid $(Pt(J), *, 0)$.

Proposition 5.1. *$(\delta, Pt(J))$ is a weak L-module.*

Proof. $\delta : L \times Pt(J) \to Pt(J)$ is defined by $\delta(a, \downarrow x) = \downarrow \sigma(a, x)$.

(i) $\delta(a+b, \downarrow x) = \delta(a \sqcup b, \downarrow x) = \downarrow \sigma(a \sqcup b, x) = \downarrow (\sigma(a,x) \vee \sigma(b,x))$
$= \downarrow \sigma(a,x) * \downarrow \sigma(b,x) = \delta(a, \downarrow x) * \delta(b, \downarrow x)$.

(ii) $\delta(a, \downarrow x * \downarrow y) = \delta(a, \downarrow x \vee y) = \downarrow \sigma(a, x \vee y) = \downarrow (\sigma(a,x) \vee \sigma(a,y))$
$= \downarrow \sigma(a,x) * \downarrow \sigma(a,y) = \delta(a, \downarrow x) * \delta(a, \downarrow y)$.

(iii) $\delta(a,0) = \downarrow \sigma(a,0) = \downarrow 0 = 0$.

(iv) $\delta(a.b, \downarrow x) = \delta(a \sqcap b, x) = \downarrow \sigma(a \sqcap b, x) = \downarrow \sigma(a, \sigma(b,x))$
$= \delta(a, \downarrow \sigma(b,x)) = \delta(a, \delta(b, \downarrow x))$.

(v) $\delta(0, \downarrow x) = \downarrow \sigma(0,x) = \downarrow 0 = 0$
$\delta(1, \downarrow x) = \downarrow \sigma(1,x) = \downarrow x$.

Hence $(\delta, Pt(J))$ is a weak L-module. ■

For each $x \in (\sigma, J)$ define $\lambda_x = \{\downarrow y \in Pt(J) : x \in \downarrow y\}$.

Proposition 5.2. *Let (σ, J) be an L-slice and $x, y \in (\sigma, J)$. Then*

(i) $\lambda_0 = Pt(J)$.
(ii) $\lambda_x \cap \lambda_y = \lambda_{x \vee y}$.

Proof. (i) $\lambda_0 = \{\downarrow y \in Pt(J) : 0 \in \downarrow y\}$. Since ideal of a slice is closed under taking lower elements, $0 \in \downarrow y$, for every $\downarrow y \in Pt(J)$. Hence $\lambda_0 = Pt(J)$.

(ii) $\downarrow z \in \lambda_x \cap \lambda_y \Rightarrow \downarrow z \in \lambda_x$ and $\downarrow z \in \lambda_y$
$\Rightarrow x \in \downarrow z$ and $y \in \downarrow z$
$\Rightarrow x \leq z$ and $y \leq z$
$\Rightarrow x \vee y \leq z$
$\Rightarrow \downarrow z \in \lambda_{x \vee y}$.

Hence $\lambda_x \cap \lambda_y \subseteq \lambda_{x \vee y}$. We have

$$\begin{aligned} \downarrow z \in \lambda_{x \vee y} &\Rightarrow x \vee y \leq z \\ &\Rightarrow x, y \leq x \vee y \leq z \\ &\Rightarrow \downarrow z \in \lambda_x \quad \text{and} \quad \downarrow z \in \lambda_y \\ &\Rightarrow \downarrow z \in \lambda_x \cap \lambda_y. \end{aligned}$$

Thus $\lambda_{x \vee y} \subseteq \lambda_x \cap \lambda_y$. Hence $\lambda_x \cap \lambda_y = \lambda_{x \vee y}$. ■

By Proposition 5.2, $B = \{\lambda_x : x \in J\}$ is closed under finite intersection and hence B is a base for a unique topology τ on Pt(J).

Proposition 5.3. *$(\delta, Pt(J), \tau)$ is a topological weak L-module.*

Proof. We have $(\delta, Pt(J))$ is a weak L-module. Let $f : Pt(J) \times Pt(J) \to Pt(J)$ be defined by $f(\downarrow x, \downarrow y) = \downarrow x * \downarrow y = \downarrow x \vee y$. We will show that f is continuous with respect to the topology τ. Let U be any open set containing $f(\downarrow x, \downarrow y) = \downarrow x * \downarrow y = \downarrow x \vee y$. Then there exists a basic open set λ_z such that $\downarrow x \vee y \in \lambda_z$ and $\lambda_z \subseteq U$. $\downarrow x \vee y \in \lambda_z$ implies that $x \vee y \geq z$. By construction λ_x, λ_y are open set containing $\downarrow x, \downarrow y$ respectively. Now we will show that $f(\lambda_x \times \lambda_y) \subseteq U$. Let $\downarrow a \in \lambda_x, \downarrow b \in \lambda_y$. Then $x \leq a, y \leq b$. $f(\downarrow a, \downarrow b) = \downarrow a * \downarrow b = \downarrow a \vee b$. But $x \leq a, y \leq b$ implies that $x \vee y \leq a \vee b$. Hence $z \leq x \vee y \leq a \vee b$ or $z \leq a \vee b$. Hence $\downarrow a \vee b \in \lambda_z$. Thus $f(\lambda_x \times \lambda_y) \subseteq \lambda_z \subseteq U$. Hence $f : Pt(J) \times Pt(J) \to Pt(J)$ is continuous with respect to the topology τ. Now we will show that for every $a \in L$ the map $\delta_a : Pt(J) \to Pt(J)$ defined by $\delta_a(\downarrow x) = \delta(a, \downarrow x) = \downarrow \sigma(a, x)$ is continuous. For any basic open set λ_x,

$$\begin{aligned}
\delta_a^{-1}(\lambda_x) &= \{\downarrow z \in Pt(J) : \delta_a(\downarrow z) = \delta(a, \downarrow z) \in \lambda_x\} \\
&= \{\downarrow z \in Pt(J) : \downarrow \sigma(a, z) \in \lambda_x\} \\
&= \{\downarrow z \in Pt(J) : x \leq \sigma(a, z) \leq z\} \\
&= \lambda_x.
\end{aligned}$$

Thus δ_a is continuous with respect to the topology τ. Hence $(\delta, Pt(J), \tau)$ is a topological weak L-module. ∎

Proposition 5.4. *If $f : (\sigma, J) \to (\mu, K)$ is an injective L-slice homomorphism, then there is a morphism $\phi : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ in the category TopWMod of topological weak L-modules.*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be an injective L-slice homomorphism. Define $\phi : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ by $\phi(\downarrow y) = \downarrow f^{-1}(y)$.

$$\begin{aligned}
\phi(\downarrow y *' \downarrow z) &= \phi(\downarrow y \vee' z) = \downarrow f^{-1}(y \vee' z) = \downarrow f^{-1}(y) \vee f^{-1}(z) \\
&= \downarrow f^{-1}(y) * \downarrow f^{-1}(z) = \phi(\downarrow y) * \phi(\downarrow z) \\
\phi(\rho(a, \downarrow x)) &= \phi(\downarrow \mu(a, x)) = \downarrow f^{-1}(\mu(a, x)) = \downarrow \sigma(a, f^{-1}(x)) \\
&= \delta(a, \downarrow f^{-1}(x)) = \delta(a, \phi(\downarrow x)).
\end{aligned}$$

Now we will show that the map ϕ is continuous. Let λ_z be an open set containing $\phi(\downarrow x) = \downarrow f^{-1}(x)$. Then $\downarrow f^{-1}(x) \in \lambda_z$ and so $z \leq f^{-1}(x)$. Thus $f(z) \leq x$ and so $\downarrow x \in \lambda_{f(z)}$. Thus $\lambda_{f(z)}$ is an open set containing $\downarrow x$. We will show that $\phi(\lambda_{f(z)}) \subseteq \lambda_z$. Let $\downarrow a \in \lambda_{f(z)}$. Then $f(z) \leq a$. Now $\phi(\downarrow a) = \downarrow f^{-1}(a)$. $f(z) \leq a$ implies that $z \leq f^{-1}(a)$. Hence $\downarrow f^{-1}(a) \in \lambda_z$. Thus $\phi(\lambda_{f(z)}) \subseteq \lambda_z$. Hence ϕ is continuous. Thus ϕ is a morphism in the category TopWMod. ∎

Remark 5.5. Let iL-slice denote the category whose objects are the collection of all L-slices and whose morphisms are all injective L-slice homomorphisms. Then iL-slice is a subcategory of L-slice.

Proposition 5.6. *There is contravariant functor from the category iL-slice to the* *category TopWMod.*

Proof. Define $\Psi : Ob(\text{iL}-slice) \to Ob(TopWMod)$ by $\Psi(J) = Pt(J)$. Also define $\Psi : Mor(\text{iL}-slice) \to Mor(TopWMod)$ as follows. Let $f : (\sigma, J) \to (\mu, K)$ be an injective L-slice homomorphism. Define $\Psi(f) : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ by $\Psi(f)(\downarrow x) = \downarrow f^{-1}(x)$. Then by above proposition $\Psi(f) \in Mor(TopWMod)$. Let $f : (\sigma, J) \to (\mu, K)$ and $g : (\mu, K) \to (\upsilon, K')$ be injective L-slice homomorphisms. We have

$$\begin{aligned} \Psi(g \circ f)(\downarrow x) &= \downarrow (g \circ f)^{-1}(x) = \downarrow f^{-1}(g^{-1}(x)) = \Psi(f)(\downarrow g^{-1}(x)) \\ &= \Psi(f)(\Psi(g)(\downarrow x)) = \Psi(g) \circ \Psi(f)(\downarrow x). \end{aligned}$$

Let $id : (\sigma, J) \to (\sigma, J)$ be an identity morphism in iL-slice. Then $\Psi(id)(\downarrow x) = \downarrow id^{-1}(x) = \downarrow x$. Hence $\Psi(id)$ is an identity morphism in TopWMod. This shows that Ψ is a contravariant functor from the category iL-slice to the category TopWMod. ■

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1994.

[2] G. Gratzer, *General Lattice Theory*, Birkhauser, 2003.

[3] H. Herrlich, G.E. Strecker, *Category Theory: An Introduction*, Allyn and Bacon, 1973.

[4] P.T. Johnstone, The point of pointless topology, *Bulletin of American Mathematical Society* **8** (1983) 41–53.

[5] P.T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[6] A.I. Kashu, Closure operator in modules and adjoint functors, I, *Algebra and Discrete mathematics* **25** (2018) 98–117.

[7] J. Picado, A. Pultr, *Frames and Locales: Topology without Points*, Front. Math., Springer, Basel, 2012.

[8] J. Picado, A. Pultr, A. Tozzi, Joins of closed sublocales, *Houston Journal of Mathematics* **45** (1) (2019) 21–38.

[9] C. Russo, *Quantale Modules with Applications to Logic and Image Processing*, Ph.D. Thesis, University of Salerno-Italy, 2007.

[10] K.S. Sabna, N.R. Mangalambal, Fixed points with respect to L-slice homomorphism σ_a, *Archivum Mathematicum* **55** (1) (2019) 43–53.

[11] K.S. Sabna, N.R. Mangalambal, Ideals and congruence with respect to frame homomorphism, *Advances in Mathematics: Scientific Journal* **9** (4) (2020) 2031–2037.

[12] M.H. Stone, Boolean algebras and their application to topology, *Proc. Nat. Acad. Sci.* **20** (1934) 197–202.

[13] M.H. Stone, The theory of representation for Boolean algebras, *Trans. Amer. Math. Soc.* **40** (1936) 37–111.

[14] M.H. Stone, Topological representation of distributive lattices and Bouwerian logics, *Casopis Pest. Mat. Fys.* **67** (1937) 1–25.