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Generalization of Condition (PWP)

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Abstract. An act A_S satisfies Condition (PWP) if $as = a's$ for $a, a' \in A_S, s \in S$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. In this paper we introduce a generalization of Condition (PWP) called Condition (L_{PWP}) and will characterize monoids using this property. It can be seen easily that any right locally cyclic act satisfies Condition (L_{PWP}).

Keywords: S-act; Condition (PWP); Locally cyclic; Condition (P).

1. Introduction

Recall that an act A_S satisfies Condition (PWP) if $as = a's$ for $a, a' \in A_S, s \in S$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. It is clear that Condition (PWP) implies torsion freeness.

Definition 1.1. *The right S-act A_S satisfies Condition (L_{PWP}) if for $a, a' \in A_S, s \in S$, the equality $as = a's$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$ and $a' = a''v$.*

The above definition is equivalent to the following statement:

$$(\forall a, a' \in A_S)(\forall s \in S)[(as = a's) \Rightarrow (\exists a'' \in A_S)(aS \cup a'S \subseteq a''S)$$

Clearly in **Act** $- S$, Condition (PWP) implies Condition (L_{PWP}) and also we have cyclic $\Longrightarrow$ locally cyclic $\Longrightarrow$ Condition (L_{PWP}).

The following example shows that Condition (L_{PWP}) does not imply Condition (PWP).

Example 1.2. Let $S = (\mathbb{N}, \cdot)$ and $K = 2\mathbb{N}$ which is a right ideal of S. Since S/K_S is cyclic, it satisfies Condition (L_{PWP}). But S/K_S is not torsion free by [6, III, Proposition 8.10]. Indeed, the element $2 \in 2\mathbb{N}$ is right cancellabe and we have $3 \cdot 2 \in 2\mathbb{N}$ but $3 \notin 2\mathbb{N}$.

Let I be a proper right ideal of a monoid S. Suppose that x, y and z are different symbols that do not belong to S. Consider the right S-act $A(I) = S\coprod^I S = ((S \setminus I) \times \{x, y\}) \cup (I \times \{z\})$ with the S-action defined by

$$(t,u)s = \begin{cases} (ts, u) & \text{if } \ ts \in S \setminus I, \\ (ts, z) & \text{if } \ ts \in I, \end{cases}$$

where $u \in \{x, y\}$ and $(t, z)s = (ts, z)$.

An act A_S is called *strongly faithful* if for $s, t \in S$, the equality $as = at$ for some $a \in A_S$, implies that $s = t$. A_S is called *faithful* if for $s, t \in S$, the equality $as = at$ for all $a \in A_S$, implies that $s = t$.

Lemma 1.3. *For the right S-act $A(I)$ the following statements holds:*

(1) *$A(I)$ does not satisfy Condition (L_{PWP}).*
(2) *$A(I)$ is not locally cyclic.*
(3) *$A(I)$ satisfies Condition (E).*
(4) *$A(I)$ is indecomposable and is generated by exactly two elements.*
(5) *$A(I)$ is faithful.*

Proof. (1). The equality $(1, x)s = (1, y)s$ holds in $A(I)$ for $s \in I$. It can be easily checked that we cannot find $a \in A(I)$ and $s_1, s_2 \in S$ such that $(1, x) = as_1, (1, y) = as_2$. So $A(I)$ does not satisfy Condition (L_{PWP}).

(2). $(1, x)$ and $(1, y)$ are elements of $A(I)$, but there is no cyclic subact of $A(I)$ which contains these two elements.

(3). Note that $A(I) = (1, x)S \cup (1, y)S$. Obviously, $(1, x)S \cong S_S \cong (1, y)S$. So $A(I)$ is the union of two subacts both of which satisfy Condition (E). Then $A(I)$ satisfies Condition (E).

(4). If $A(I)$ is decomposable, then there exist $B_S, C_S \leq A(I)$ such that $A(I) = B_S \cup C_S, B_S \cap C_S = \emptyset$. Let $(1, x) \in B_S$. Since $A(I) = (1, x)S \cup (1, y)S$, we have $(1, y) \in C_S$. But $(1, x)S \cap (1, y)S = I \subseteq B_S \cap C_S$ which shows that $B_S \cap C_S \neq \emptyset$, a contradiction. Hence $A(I)$ is indecomposable.

The equality $A(I) = (1, x)S \cup (1, y)S$ shows that $A(I)$ is generated by exactly two elements.

(5). It was mentioned that $A(I) = (1, x)S \cup (1, y)S$ and $(1, x)S \cong S_S \cong (1, y)S$. Since S_S is faithful, $A(I)$ has a faithful subact. Therefore, $A(I)$ is faithful. ∎

2. Classification of Monoids by Condition (L_{PWP}) of their Acts

Recall that an act A_S satisfies Condition (E') if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = ut$. A_S satisfies Condition $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$.

Theorem 2.1. *The following statements are equivalent:*

(1) *All right S-acts satisfy Condition* (L_{PWP}).
(2) *All right S-acts satisfying Condition* $(E'P)$*, satisfy Condition* (L_{PWP}).
(3) *All right S-acts satisfying Condition* (EP)*, satisfy Condition* (L_{PWP}).
(4) *All right S-acts satisfying Condition* (E')*, satisfy Condition* (L_{PWP}).
(5) *All right S-acts satisfying Condition* (E)*, satisfy Condition* (L_{PWP}).
(6) *All faithful right S-acts satisfy Condition* (L_{PWP}).
(7) *All indecomposable right S-acts satisfy Condition* (L_{PWP}).
(8) *S is a group.*

Proof. Since $(E)\Rightarrow(E')\Rightarrow(E'P)$ and $(E)\Rightarrow(EP)\Rightarrow(E'P)$, the implications (1)⇒(2)⇒(4)⇒(5), (2)⇒(3)⇒(5), (1)⇒(6) and (1)⇒(7) are obvious.

(5)⇒(8). Take $s \in S$ such that $sS \neq S$. Lemma 1.3 implies that $A(sS)$ satisfies Condition (E). By assumption, $A(sS)$ satisfies Condition (L_{PWP}) which is not possible by Lemma 1.3. Thus $sS = S$ for any $s \in S$ and S is a group.

(8) ⇒ (1). If S is a group, then any right S-act satisfies Condition (L_{PWP}). To see this, let A_S be an arbitrary right S-act such that for $a, a' \in A_S$ and $s \in S$ we have $as = a's$. Then we can write $a = a'ss^{-1}$ and $a' = a'.1$. Thus A_S satisfies Condition (L_{PWP}).

(6) ⇒ (8). Take $s \in S$ such that $sS \neq S$. By Lemma 1.3, $A(sS)$ is faithful, so by assumption $A(sS)$ satisfies Condition (L_{PWP}) which is impossible by Lemma 1.3. Then for any $s \in S, sS = S$ which means that S is a group.

(7) ⇒ (8). Let $s \in S$ be such that $sS \neq S$. The right S-act $A(sS)$ is indecomposable by Lemma 1.3. So $A(sS)$ should satisfy Condition (L_{PWP}) and this is impossible. Hence $sS = S$ for any $s \in S$ which shows that S is a group. ■

As we see in Example 1.2, Condition(L_{PWP}) does not imply torsion freeness.

Theorem 2.2. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition* (L_{PWP})*, are torsion free.*
(2) *All finitely generated right S-acts satisfying Condition* (L_{PWP}) *are torsion free.*
(3) *All right S-acts generated by at most two elements satisfying Condition* (L_{PWP}) *are torsion free.*
(4) *All right cancellable elements are right invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By assumption any cyclic right S-act satisfies Condition (L_{PWP}), is torsion free, and so by [6, IV, Theorem 6.1] every right cancellable element is right invertible.

(4) $\Rightarrow$ (1). If (4) holds, then by [6, IV, Theorem 6.1] all right S-acts are torsion free, and so any right S-act satisfying Condition (L_{PWP}) is torsion free. ■

Condition (PWP_E) is defined in [2].

Theorem 2.3. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}), satisfy Condition (PWP_E).*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}), satisfy Condition (PWP_E).*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}), satisfy Condition (PWP_E).*
(4) *All right S-acts satisfying Condition (L_{PWP}) are principally weakly flat.*
(5) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are principally weakly flat.*
(6) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are principally weakly flat.*
(7) *S is a regular monoid.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. By [2, Theorem 2.3] Condition (PWP_E) implies principal weak flatness, and so we have the implications (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7). Since all cyclic right S-acts satisfy Condition (L_{PWP}), by assumption all cyclic right S-acts are principally weakly flat. Then by [6, IV, Theorem 6.6] S is regular.

(7) $\Rightarrow$ (1). By [2, Theorem 3.1], all right S-acts satisfy Condition (PWP_E) and consequently all right S-acts satisfying Condition (L_{PWP}) satisfy Condition (PWP_E). ■

Condition (P_E) was introduced in [4].

Theorem 2.4. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}), satisfy Condition (P_E).*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}), satisfy Condition (P_E).*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}), satisfy Condition (P_E).*
(4) *All right S-acts satisfying Condition (L_{PWP}) are weakly flat.*

(5) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are weakly flat.*
(6) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are weakly flat.*
(7) *S is regular and satisfies the following condition:*
 (R) *For any elements $s, t \in S$, there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$ ($\rho(s,t)$ is the smallest right congruence on S containing (s,t)).*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. By [4, Theorem 2.3] Condition (P_E) implies weak flatness, and so we have the implications (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7). Since all cyclic right S-acts satisfy Condition (L_{PWP}), by assumption all cyclic right S-acts are weakly flat. Then by [6, IV, Theorem 7.5] S is regular and satisfies Condition (R).

(7) $\Rightarrow$ (1). By [6, IV, Theorem7.5] all right S-acts are weakly flat. It follows from [4, Theorem 2.5] that all right S-acts satisfy Condition (P_E), and so all right S-acts satisfying Condition (L_{PWP}) satisfy Condition (P_E). ■

Theorem 2.5. *The following statements are equivalent:*
(1) *All right S-acts satisfying Condition (L_{PWP}), satisfy Condition (P).*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}), satisfy Condition (P).*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}), satisfy Condition (P).*
(4) *S is a group or a group with a zero adjoined.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since all cyclic right S-acts satisfy Condition (L_{PWP}), by assumption we can deduce that all cyclic right S-acts satisfy Condition (P). By [8, Theorem 2.1], $S = G^0$ or $S = G$, where G is a group.

(4) $\Rightarrow$ (1). If $S = G$, then by [9, Theorem 3.10] all right S-acts satisfy Condition (P). Therefore, all right S-acts satisfying Condition (L_{PWP}), satisfy Condition (P). Suppose that $S = G^0$ and let A_S be a right S-act which satisfies Condition (L_{PWP}). We show that A_S satisfies Condition (P). Take $a, a' \in A_S$ and $s, t \in S$ such that $as = a't$. There are three cases as follows:

Case 1: If $s, t \in G$, then we have $a = a'ts^{-1}, a' = a'1$ and $(ts^{-1})s = 1t$.

Case 2: If $t = 0, s \in G$, then $as = a'0$ implies that $(as)s^{-1} = (a'0)s^{-1}$. So $a = a'0, a' = a'1$ and $0s = 10$.

Case 3: If $s = t = 0$, then $a0 = a'0$. Since A_S satisfies Condition (L_{PWP}), there exist $a'' \in A_S, u, v \in S$ such that $a = a''u, a' = a''v$ and clearly $u0 = v0$. ■

We deduce from the above Theorem that whenever $S \neq G^0$ and $S \neq G$ (G is a group), then there exists at least one (finitely generated) right S-act that satisfies Condition (L_{PWP}) but does not satisfy Condition (P).

Recall that a monoid S is called left collapsible if for every $s, s' \in S$, there exists $u \in S$ such that $us = us'$. S is called weakly left collapsible if for every $s, s', z \in S$, $sz = s'z$ implies the existence of $u \in S$ such that $us = us'$.

Example 2.6. Let S be a monoid which is not weakly left collapsible. Then the one-element right S-act $\Theta_S = \{\theta\}$ with multiplication $\theta s = \theta$ for all $s \in S$, fails to satisfy Condition (E') but it satisfies Condition (L_{PWP}). So Condition (L_{PWP}) does not imply Condition (E') (as well as Condition (E), regularity, strong faithfulness, equalizer flatness which are stronger properties).

Theorem 2.7. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}), satisfy Condition (E').*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}), satisfy Condition (E').*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}), satisfy Condition (E').*
(4) $(\forall s, t, z \in S), (sz = tz \Longrightarrow (\exists e \in E(S))(\rho(s, t) = ker\lambda_e))$

Proof. (1) ⇒ (2) ⇒ (3). It is clear.

(3) ⇒ (4). Since all cyclic right S-acts satisfy Conditon (L_{PWP}), by assumption all cyclic right S-acts satisfy Condition (E'). By [3, Theorem 2.5] the result follows.

(4) ⇒ (1). By [3, Theorem 2.5], all right S-acts satisfy Condition (E'). Hence all right S-acts satisfying Condition (L_{PWP}), satisfy Condition (E'). ■

Weak pullback flatness was defined by Laan [7]. He showed that a right S-act is pullback flat if and only if it satisfies both Conditions (P) and (E').

Theorem 2.8. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}) are weakly pullback flat.*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are weakly pullback flat.*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are weakly pullback flat.*
(4) *S is a group or $S = \{0, 1\}$.*

Proof. (1) ⇒ (2) ⇒ (3). It is obvious.

(3) ⇒ (4). By assumption, since all cyclic right S-acts satisfy Condition (L_{PWP}), one can deduce that all cyclic right S-acts are weakly pullback flat. So by [1, Proposition 25] the statement (4) holds.

(4) ⇒ (1). If S is a group, then by [1, Proposition 9] all right S-acts are weakly pullback flat and we are done. Now suppose that $S = \{0, 1\}$. Let A_S be a right S-act that satisfies Condition (L_{PWP}). It suffices to show that A_S

satisfies Conditions (P) and (E'). Let $a, a' \in A_S$ and $s, t \in S$ be such that $as = a't$. To show that A_S satisfies Condition (P), we consider three cases:

Case 1: $s = t = 1$. Then we have $a = a'$, and so we can write $a = a' \cdot 1, a' = a' \cdot 1$ and $1 \cdot s = 1 \cdot t$.

Case 2: $s = 1, t = 0$. Then we have $a = a' \cdot 0, a' = a' \cdot 1$ and $0 \cdot s = 1 \cdot t$.

Case 3: $s = t = 0$. Since A_S satisfies Condition (L_{PWP}), the equality $a0 = a'0$ implies that there exist $a'' \in A_S, u, v \in S$ such that $a = a''u \wedge a = a''v$ and clearly $u \cdot 0 = v \cdot 0$.

So A_S satisfies Condition (P) in each case. It remains to show that A_S satisfies Condition (E'). Since $S = \{0, 1\}$, by [9, Theorem 2.4] all cyclic right S-acts satisfy Condition (E), and consequently satisfy Condition (E'). ■

Theorem 2.9. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}) are strongly flat.*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are strongly flat.*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are strongly flat.*
(4) *All right S-acts satisfying Condition (L_{PWP}) are equalizer flat.*
(5) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are equalizer flat.*
(6) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are equalizer flat.*
(7) *All right S-acts satisfying Condition (L_{PWP}), satisfy Condition (E).*
(8) *All finitely generated right S-acts satisfying Condition (L_{PWP}), satisfy Condition (E).*
(9) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}), satisfy Condition (E).*
(10) $S = \{0, 1\}$ *or* $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (8) $\Rightarrow$ (9) are obvious.

Since we have strongly flat $\Rightarrow$ equalizer flat $\Rightarrow$ Condition (E), the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (7) and (3) $\Rightarrow$ (6) $\Rightarrow$ (9) are valid.

(9) $\Rightarrow$ (10). Since all cyclic right S-acts satisfy Condition (L_{PWP}), by assumption, all cyclic right S-acts satisfy Condition (E). By [9, Theorem 2.4] we are done.

(10) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are strongly flat and (1) follows. Suppose that $S = \{0, 1\}$. Then by Theorem 2.8 all right S-acts satisfying Condition (L_{PWP}), are weakly pullback flat which is the combination of Conditions (E') and (P). Since $S = \{0, 1\}$, Condition (E') implies Condition (E). So, all right S-acts satisfying Condition (L_{PWP}), are strongly flat. ■

Recall that a monoid S satisfies Condition (K) if, every left collapsible submonoid of S contains a left zero. The monoid S satisfies Condition A if any right S-act satisfy the ascending chain condition for cyclic subacts.

Theorem 2.10. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}) are projective.*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are projective.*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are projective.*
(4) $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since projectivity implies strong flatness, by assumption all right S-acts generated by at most two elements satisfying Condition (L_{PWP}), are strongly flat. So by [8, Corollary 2.2], $S = \{1\}$ or $S = \{0, 1\}$.

(4) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are projective. Now suppose that $S = \{0, 1\}$. By Theorem 2.9, all right S-acts satisfying Condition (L_{PWP}) are strongly flat. Assume that A_S is a right S-act and $\alpha_1 S \subseteq \alpha_2 S \subseteq ...$ be an ascending chain of cyclic subacts of A_S. Since $S = \{0, 1\}$, we have for all $i \in \mathbb{N}, 1 \leq\mid \alpha_i S \mid\leq 2$. If $\mid \alpha_i S \mid= 1$ for all $i \in \mathbb{N}$, then $\alpha_i S = \alpha_j S$, for every $i, j \in \mathbb{N}$ and we are done. Suppose that $\mid \alpha_k S \mid= 2$ for some $k \in \mathbb{N}$. So $\mid \alpha_i S \mid= 2$ for every $i \geq k$. This shows that for all $i \geq k, \alpha_i S = \alpha_{i+1} S$, which means that the above chain terminates. Thus the monoid S satisfies Condition (A). Obviously, the monoid S satisfies Condition (K), and so by Theorems 1.1 and 2.1 of [5], strong flatness implies projectivity. ∎

Theorem 2.11. *The following statements are equivalent:*

(1) *All right S-acts satisfying Condition (L_{PWP}) are free.*
(2) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are free.*
(3) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are free.*
(4) *All right S-acts satisfying Condition (L_{PWP}) are projective generator.*
(5) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are projective generator.*
(6) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are projective generator.*
(7) *All right S-acts satisfying Condition (L_{PWP}) are generator.*
(8) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are generator.*
(9) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are generator.*
(10) *All right S-acts satisfying Condition (L_{PWP}) are strongly faithful.*

(11) *All finitely generated right S-acts satisfying Condition (L_{PWP}) are strongly faithful.*
(12) *All right S-acts generated by at most two elements satisfying Condition (L_{PWP}) are strongly faithful.*
(13) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12), are obvious.

Since free $\Rightarrow$ projective generator $\Rightarrow$ generator, the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (7) and (3) $\Rightarrow$ (6) $\Rightarrow$ (9) are easily obtained.

(12) $\Rightarrow$ (13). Since the one element act Θ_S satisfy Condition (L_{PWP}), by assumption Θ_S is strongly faithful. So $S = \{1\}$.

(13) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are free.

(13) $\Rightarrow$ (10). If $S = \{1\}$, then all right S-acts are strongly faithful.

(9) $\Rightarrow$ (13). By assumption Θ_S is generator. So there exists an epimorphism $\pi : \Theta_S \to S_S$ and this implies that $S = \{1\}$. ■

For fixed elements $u, v \in S$, define a binary relation $P_{u,v}$ on S with

$$(x, y) \in P_{u,v} \iff ux = vy \ (x, y \in S).$$

Recall that for any right ideal I of S, Rees congruence ρ_I on S defined by $(x, y) \in \rho_I$ if $x = y$ or $x, y \in I$.

Recall that an act is called cofree if it is isomorphic to the act $X^S = \{f | f$ is a mapping from S to $X\}$ for some nonempty set X, where fs is defined by $fs(t) = f(st), t \in S$, for every $f \in X^S, s \in S$.

Theorem 2.12. *The following conditions are equivalent:*

(1) *All fg-weakly injective right S-acts satisfy Condition (L_{PWP}).*
(2) *All weakly injective right S-acts satisfy Condition (L_{PWP}).*
(3) *All injective right S-acts satisfy Condition (L_{PWP}).*
(4) *All cofree right S-acts satisfy Condition (L_{PWP}).*
(5) *For any $s \in S$ there exist $u, v \in S$ such that:*
   (i) $P_{u,v} \subset P_{1,s} \circ ker\lambda_s \circ P_{s,1}$.
   (ii) $ker\lambda_u \cup ker\lambda_v \subset \rho_{sS}$.
   (iii) $ker\lambda_{us} \cup ker\lambda_{vs} \subset ker\lambda_s$

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5). Suppose that $s \in S$ and S_1 and S_2 are two sets such that $|S_1| = |S| = |S_2|$. Assume that $\alpha : S \to S_1$ and $\beta : S \to S_2$ are bijections. Let $X = S/ker\lambda_s \coprod S_1 \coprod S_2$. Define the mappings $f, g : S \longrightarrow X$ as follows:

$$f(x) = \begin{cases} [y]_{ker\lambda_s} & \text{if } \ x \in sS (x = sy), \\ \alpha(x) & \text{if } \ x \notin sS \end{cases}$$

and

$$g(x)=\begin{cases}[y]_{ker\lambda_s} & \text{if } x\in sS (x=sy),\\ \beta(x) & \text{if } x\notin sS.\end{cases}$$

If $y_1,y_2\in S$ are such that $sy_1=sy_2$, then $(y_1,y_2)\in ker\lambda_s$ which implies that $f(sy_1)=[y_1]_{ker\lambda_s}=[y_2]_{ker\lambda_s}=f(sy_2)$. So f is well-defined. By a similar argument, g is well-defined.

Clearly $fs=gs$ by definition. Since X^S satisfies Condition (L_{PWP}), there exist $u,v\in S$ and the map $h:S\to X$ such that $g=hv$ and $f=hu$. Now, we show that the conditions (i)–(iii) are true.

(i). Let $(l_1,l_2)\in P_{u,v}$, $l_1,l_2\in S$. Then $ul_1=vl_2$, and so $f(l_1)=hu(l_1)=h(ul_1)=h(vl_2)=hv(l_2)=g(l_2)$. The equality $f(l_1)=g(l_2)$ yields that there exist $y_1,y_2\in S$ such that $l_1=sy_1, l_2=sy_2, (y_1,y_2)\in ker\lambda_s$. So $(l_1,y_1)\in P_{1,s}, (y_1,y_2)\in ker\lambda_s$ and $(y_2,l_2)\in P_{s,1}$. This implies that $(l_1,l_2)\in P_{1,s}\circ ker\lambda_s\circ P_{s,1}$.

(ii). Assume that the condition (ii) does not hold. Without loss of generality, suppose that $ker\lambda_u\setminus\rho_{sS}\neq\emptyset$. So there exist $p_1,p_2\in S$ such that $(p_1,p_2)\in ker\lambda_u\setminus\rho_{sS}$. That is

$$(\exists p_1,p_2\in S)(up_1=up_2\wedge p_1\neq p_2\wedge(p_1\notin sS\vee p_2\notin sS)).$$

Hence $f(p_1)=hu(p_1)=h(up_1)=h(up_2)=hu(p_2)=f(p_2)$. Since $p_1\neq p_2$ and α is injective, $p_1,p_2\in sS$ which is a contradiction.

(iii). Suppose that $(l_1,l_2)\in ker\lambda_{us}$ where $l_1,l_2\in S$. So $usl_1=usl_2$ and we have $f(sl_1)=hu(sl_1)=h(usl_1)=h(usl_2)=hu(sl_2)=f(sl_2)$. According to the definition of f, we have $sl_1=sl_2$ or $[l_1]_{ker\lambda_s}=[l_2]_{ker\lambda_s}$. If $sl_1=sl_2$, then $(l_1,l_2)\in ker\lambda_s$. If $[l_1]_{ker\lambda_s}=[l_2]_{ker\lambda_s}$, then $(l_1,l_2)\in\ker\lambda_s$. So $ker\lambda_{us}\subseteq\ker\lambda_s$. Similar argument shows that $ker\lambda_{vs}\subseteq ker\lambda_s$. Consequently $ker\lambda_{us}\cup ker\lambda_{vs}\subseteq ker\lambda_s$.

(5) $\Rightarrow$ (1). Suppose that A_S is fg-weakly injective and for $a,a'\in A_S, s\in S, as=a's$. By assumption there exist $u,v\in S$ such that the conditions (i)–(iii) hold. Define $\varphi:uS\cup vS\to A_S$ by the rule that for each x in $uS\cup vS$,

$$\varphi(x)=\begin{cases}ap & \text{if } x\in uS(x=up),\\ a'r & \text{if } x\in vS(x=vr).\end{cases}$$

To show that φ is well defined, we consider three cases as follows:

Case 1. If there exist $p,r\in S$ such that $up=vr$, then by condition (i), there exist $y_1,y_2\in S$ such that $(p,y_1)\in P_{1,s}, (y_1,y_2)\in ker\lambda_s, (y_2,r)\in P_{s,1}$. Thus $p=sy_1, r=sy_2$ and $sy_1=sy_2$, and so $ap=asy_1=a'sy_1=a'sy_2=a'r$.

Case 2. Assume that there exist $p_1,p_2\in S$ such that $up_1=up_2$. If $p_1=p_2$, then $ap_1=ap_2$. If $p_1\neq p_2$, then by condition (ii), there exist $y'_1,y'_2\in S$ such that $p_1=sy'_1$ and $p_2=sy'_2$. Then by (iii) we have

$$up_1=up_2\Rightarrow usy'_1=usy'_2\Rightarrow(y'_1,y'_2)\in ker\lambda_{us}\subseteq ker\lambda_s.$$

This means that $sy'_1=sy'_2$. Hence $ap_1=asy'_1=asy'_2=ap_2$.

Case 3. If there exist $r_1, r_2 \in S$ such that $vr_1 = vr_2$, then by a similar argument we get $a'r_1 = a'r_2$. Hence φ is well defined. Clearly φ is a homomorphism. Since A_S is fg-weakly injective, there exists a homomorphism $\psi : S_S \to A_S$ which extends φ. Put $a'' = \psi(1)$. Then $a = \varphi(u) = \psi(u) = \psi(1)u = a''u$ and $a' = \varphi(v) = \psi(v) = \psi(1)v = a''v$. Thus A_S satisfies Condition (L_{PWP}). ■

Corollary 2.13. *If S is a commutative monoid, then all cofree right S-acts satisfy Condition (L_{PWP}) if and only if S is a group.*

Proof. Let $s \in S$. By Theorem 2.12, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$. If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which implies that $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Commutativity of S implies that $(v, u) \in P_{u,v}$ which yeilds $u = v$. So $ker\lambda_u = P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}$. Since $u.1 = u.1$, there exist $y_1, y_2 \in S$ such that $(1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, 1) \in P_{s,1}$. Hence $1 = sy_1 = sy_2$ and S is a group.

Conversely, suppose that S is a group. By Theorem 2.1, all right S-acts satisfy Condition (L_{PWP}). So all right cofree right S-acts satisfy Condition (L_{PWP}). ■

Corollary 2.14. *If S is a finite monoid, then all cofree right S-acts satisfy Condition (L_{PWP}) if and only if S is a group.*

Proof. By Theorem 2.12, for any $s \in S$ there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,s} \circ ker\lambda_s \circ P_{s,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{sS}$ and $ker\lambda_{us} \cup ker\lambda_{vs} \subseteq ker\lambda_s$. If $l_1, l_2 \in S$ and $ul_1 = vl_2$, then there exist $y_1, y_2 \in S$ such that $(l_1, y_1) \in P_{1,s}, (y_1, y_2) \in ker\lambda_s, (y_2, l_2) \in P_{s,1}$ which means $l_1 = sy_1, sy_1 = sy_2, sy_2 = l_2$. So $l_1 = l_2$ and $P_{u,v} \subseteq \Delta_S$. Suppose that $l_1, l_2 \in S$ are such that $ul_1 = ul_2$ and $l_1 \neq l_2$. Then there exist $y_1, y_2 \in S, l_1 = sy_1, l_2 = sy_2$ which implies that $usy_1 = usy_2$. The last equality shows that $(y_1, y_2) \in ker\lambda_{us} \subseteq ker\lambda_s$. So $sy_1 = sy_2$ that is $l_1 = l_2$ which is a contradiction. Hence u is left cancellable. Let $S = \{1, x_1, x_2, ..., x_n\}$ (note that the elements of S are distinct). It is clear that $uS = \{u, ux_1, ux_2, ..., ux_n\} = S$. So $v \in uS$. If there exists $i \leq n$ such that $ux_i = v$, then $(x_i, 1) \in P_{u,v} \subseteq \Delta_S$ which implies that $x_i = 1$, a contradiction. Hence $v = u$ and by a similar argument to Corollary 2.13, we get that s has a right inverse. Thus S is a group.

The converse has been proved in Theorem 2.1. ■

Corollary 2.15. *If S is an idempotent monoid, then all cofree right S-acts satisfy Condition (L_{PWP}) if and only if $S = \{1\}$.*

Proof. Suppose that all cofree S-acts satisfy Condition (L_{PWP}). We claim that $S = \{1\}$. Assume that $S \neq \{1\}$. So there exists $e \in S \setminus \{1\}$. By Theorem 2.12, there exist $u, v \in S$ such that $P_{u,v} \subseteq P_{1,e} \circ ker\lambda_e \circ P_{e,1}, ker\lambda_u \cup ker\lambda_v \subseteq \rho_{eS}$. Obviously, $(u, 1) \in ker\lambda_u \subseteq \rho_{eS}$, so $u = 1$ or there exist $y_1, y_2 \in S$ such that $u = ey_1$ and $1 = ey_2$. Since $e \neq 1$, we get that $u = 1$ and similarly $v = 1$. By a similar argument to the Corollary 2.13, e has a right inverse. Hence $e = 1$ which is a contradiction. Thus $S = \{1\}$ and we are done.

The converse is a part of Theorem 2.1. ■

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